1) (a) Population at end of year 2

$= 50\,000 \times 1.025 = 51,250$

(b) Population at end of year N

$= 50\,000 \, (1.025)^{N-1}$

so $50000 \, (1.025)^{N-1} > 500\,000$

$(1.025)^{N-1} > \frac{500\,000}{50\,000}$

$(1.025)^{N-1} > 10$

taking $\log_{10}$ of both sides

$(N-1) \log_{10} (1.025) > \log 10 = 1$

$(N-1) \log 1.025 > 1$

(c) $N-1 > \frac{1}{\log(1.025)}$

$N > 1 + \frac{1}{\log(1.025)}$

$N > 94$

So at end of year 94 or in year 95 the population will exceed 500 000.

1 (d) Let $D_N$ be the donation at end of year N

$$D_1 = 50\,000 \times 0.50 = 25\,000$$

$$D_N = 50000\,(1.025)^{N-1} \times 0.5$$
$$= 25000\,(1.025)^{N-1}$$

let $S_N$ be the sum of donations

this is a g.p with $a = 25000$
$r = 1.025$

$$S_{20} = \frac{25000\,(1 - 1.025^{20})}{1 - 1.025} = \$638,616$$

$$\approx \$\ 637\,000$$

2 $\dfrac{r+1}{r+2} - \dfrac{r}{r+1}$

$$\frac{(r+1)(r+2) - r((r+2)}{(r+2)(r+1)} = \frac{r^2+2r+1-r^2-2r}{(r+1)\,(r+2)} = \frac{1}{(r+1)(r+2)}$$

(ii) $\dfrac{1}{6} + \dfrac{1}{12} + \dfrac{1}{20} + \cdots \dfrac{1}{(n+1)(n+2)}$

$$= \left(\cancel{\tfrac{2}{3}} - \tfrac{1}{2}\right) + \left(\cancel{\tfrac{3}{4}} - \cancel{\tfrac{2}{3}}\right) + \left(\cancel{\tfrac{4}{5}} - \cancel{\tfrac{3}{4}}\right) + \cdots \left(\cancel{\tfrac{n}{n+1}} - \cancel{\tfrac{n-1}{n}}\right)$$
$$+ \left(\tfrac{n+1}{n+2} - \cancel{\tfrac{n}{n+1}}\right)$$

$$= -\frac{1}{2} + \frac{n+1}{n+2} = \frac{n}{2(n+2)}$$

(iii) $\displaystyle\sum_{1}^{\infty} \frac{1}{(r+1)(r+2)} = \lim_{n\to\infty} \frac{n}{2(n+2)}$

$$= \lim_{n\to\infty} \left[\frac{n+1}{n+2} - \frac{1}{2}\right]$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

3 Let $P(n)$ be the statement that

$$u_n = (5^{n-1}) + 1 \quad \text{for } n \geq 1$$

given that $u_1 = 2$ and $u_{n+1} = 5u_n - 4$

For $n = 1$ $u_1 = 2$ and $5^{1-1} + 1 = 5^0 + 1 = 1 + 1 = 2$
so $P(1)$ is true

Assume for $n = k$, $u_k = 5^{k-1} + 1$

now for $n = k+1$

$$u_{k+1} = 5u_k - 4$$

$$= 5(5^{k-1} + 1) - 4$$

$$= 5^k + 1 = 5^{(k+1)-1} + 1$$

So $P(k+1)$ is true whenever $P(k)$ is true

as since $P(1)$ is true then by M.I. $P(n)$ is true for $\forall\, n \in \mathbb{N}$, $n \geq 1$

4) $f(x) = \frac{1}{2}x^4 - x^3 + x - 3$

Since $f(x)$ is a polynomial, $f(x)$ is continuous and

since $f(2) = -1$

$f(2.5) = 3.40625$

then by intemediate value theorem is at least one root in $[2, 2.5]$

(5) $f(x) = \frac{1}{2}x^4 - x^3 + x - 3$

$f'(x) = 2x^3 - 3x^2 + 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_1 = 2.25$

$$x_2 = 2.25 - \frac{f(2.25)}{f'(2.25)}$$

$f(2.25) = 0.673828125$

$f'(2.25) = 8.59375$

$$x_2 = 2.25 - \frac{0.673828125}{8.59375} = 2.17$$

(c) $f(x) = \frac{1}{2}x^4 - x^3 + x - 3$

$f(2) = -1$

$f(2.5) = 3.40625$

so next approximation $x = 2.25$

$f(2.25) = 0.67$

so next approximation is half of the interval $[2, 2.25] = 2.125$

to 3 sig figs $\approx 2.13$

5) $\sqrt[3]{8-9x} = \sqrt[3]{8\left(1-\frac{9}{8}x\right)}$

$= \sqrt[3]{8} \times \sqrt[3]{1-\frac{9}{8}x}$

$= 2\left(1-\frac{9}{8}x\right)^{1/3}$

$= 2\left[1 + \left(\frac{1}{3}\right)\left(-\frac{9}{8}x\right) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(-\frac{9}{8}x\right)^2\right.$

$\left. + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\left(-\frac{9}{8}x\right)^3\right.$

$= 2\left[1 - \frac{9}{24}x - \frac{9}{64}x^2 - \frac{45}{512}x^3 + \cdots\right]$

$= 2 - \frac{9}{12}x - \frac{9}{32}x^2 - \frac{45}{256}x^3 + \cdots$

$$8000 - 900 = 7100$$

$$1000(8 - 0.9) = 7100$$

$$(7100)^{1/3} = 1000^{1/3}(8 - 0.9)^{1/3}$$

$$= 10(8 - 0.9)^{1/3}$$

let $9x = 0.9 \Rightarrow x = \frac{1}{10}$ or $0.1$

So $(7100)^{1/3} = 10\left[2 - \frac{3}{4}(0.1) - \frac{9}{32}(0.1)^2 - \frac{45}{256}(0.1)^3\right]$

$$= 19.220117 \approx 19.2201$$

6) $y^2 = \sec x + \tan x$

differentiating implicitly

$$2y\frac{dy}{dx} = \tan x \sec x + \sec^2 x$$

$$\frac{dy}{dx} = \frac{1}{2y}\sec x\,(\tan x + \sec x)$$

$$= \frac{1}{2y}\sec x \quad y^2 = \frac{y}{2}\sec x \;//$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}y\,(\sec x \tan x) + \frac{1}{2}\sec x\frac{dy}{dx}$$

$$= \frac{1}{2}y \sec x \tan x + \frac{1}{2}\sec x\,\frac{1}{2}y\sec x$$

$$= \frac{1}{4}y\sec x\,(2\tan x + \sec x) \;//$$

Maclaurin's expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$f(0) = 1 \qquad f'(0) = \frac{1}{2}(1)(1) = \frac{1}{2}$$

$$f''(0) = \frac{1}{4}(1)(1)(0+1) = 1/4$$

$$\text{So } f(x) = 1 + \frac{1}{2}x + \frac{\frac{1}{4}}{2!}x^2$$

$$= 1 + \frac{1}{2}x + \frac{1}{8}x^2$$