FORM TP 2015268

TEST CODE **02234020**

MAY/JUNE 2015

**CARIBBEAN EXAMINATIONS COUNCIL**

**CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®**

# PURE MATHEMATICS

## UNIT 2 – Paper 02

## ANALYSIS, MATRICES AND COMPLEX NUMBERS

***2 hours 30 minutes***

**27 MAY 2015 (p.m.)**

**READ THE FOLLOWING INSTRUCTIONS CAREFULLY.**

1. This examination paper consists of THREE sections.
2. Answer ALL questions from the THREE sections.
3. Each section consists of TWO questions.
4. Write your solutions, with full working, in the answer booklet provided.
5. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.

**<u>Examination Materials Permitted</u>**

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable, electronic calculator

**DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.**

## SECTION A

### Module 1

**Answer BOTH questions.**

**1.** (a) Three complex numbers are given as

$$z_1 = 1 + (7 - 4\sqrt{3})i, \quad z_2 = \sqrt{3} + 3i \quad \text{and} \quad z_3 = -2 + 2i.$$

(i) Express the quotient $\frac{z_3}{z_2}$ in the form $x + iy$ where $x, y \in \mathrm{R}$. **[3 marks]**

(ii) Given that $\arg w = \arg z_3 - [\arg z_1 + \arg z_2]$, $|z_1| = 1$ and $\arg z_1 = \frac{\pi}{12}$ rewrite

$w = \frac{z_3}{z_1 z_2}$ in the form $re^{i\theta}$ where $r = |w|$ and $\theta = \arg w$. **[6 marks]**

(b) A complex number $v = x + iy$ is such that $v^2 = 2 + i$. Show that

$$x^2 = \frac{2 + \sqrt{5}}{2}.$$ **[7 marks]**

(c) The function $f$ is defined by the parametric equations

$$x = \frac{e^{-t}}{\sqrt{1 - t^2}} \quad \text{and} \quad y = \sin^{-1} t \quad \text{for } -1 < t \leq 0.5.$$

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^t(1 - t^2)}{t^2 + t - 1}$. **[6 marks]**

(ii) Hence, show that $f$ has no stationary value. **[3 marks]**

**Total 25 marks**

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**2.** (a) Let $4x^2 + 3xy^2 + 7x + 3y = 0$.

(i) Use implicit differentiation to show that

$$\frac{dy}{dx} = \frac{8x + 3y^2 + 7}{3(1 + 2xy)}.$$

**[5 marks]**

(ii) Show that for $f(x, y) = 4x^2 + 3xy^2 + 7x + 3y$

$$6\,\frac{\partial f(x, y)}{\partial y} - 10 = \left(\frac{\partial^2 f(x, y)}{\partial y^2}\right)\left(\frac{\partial^2 f(x, y)}{\partial y\,\partial x}\right) + \frac{\partial^2 f(x, y)}{\partial x^2}.$$

**[5 marks]**

(b) The rational function

$$f(x) = \frac{18x^2 + 13}{9x^2 + 4}$$

is defined on the domain $-2 \leq x \leq 2$.

(i) Express $f(x)$ in the form $a + \dfrac{b}{9x^2 + 4}$ where $a, b \in \mathrm{R}$. **[2 marks]**

(ii) Given that $f(x)$ is symmetric about the $y$-axis, evaluate $\int_{-2}^{2} f(x)\,dx$. **[6 marks]**

(c) Let $h$ be a function of $x$.

(i) Show that

$$\int h^n \ln h\, dh = \frac{h^{n+1}}{(n+1)^2}\,[-1 + (n+1)\ln h] + C,$$

where $-1 \neq n \in \mathrm{Z}$ and $C \in \mathrm{R}$. **[5 marks]**

(ii) Hence, find

$$\int \sin^2 x \cos x \ln(\sin x)\, dx.$$

**[2 marks]**

**Total 25 marks**

## SECTION B

### Module 2

**Answer BOTH questions.**

**3.** (a) The $n$th term of a sequence is given by

$$T_n = \frac{2n+1}{\sqrt{n^2+1}}.$$

(i) Determine $\lim\limits_{n \to \infty} T_n$. **[5 marks]**

(ii) Show that $T_4 = \frac{9}{4}\left(1 + \frac{1}{16}\right)^{-\frac{1}{2}}$. **[3 marks]**

(iii) Hence, use the binomial expansion with $x = \frac{1}{16}$ to approximate the value of $T_4$ for terms up to and including $x^3$. Give your answer correct to two decimal places. **[4 marks]**

(b) A series is given as

$$2 + \frac{3}{4} + \frac{4}{9} + \frac{5}{16} + \ldots$$

(i) Express the $n^{\text{th}}$ partial sum $S_n$ of the series in sigma notation. **[2 marks]**

(ii) Hence, given that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges to $\frac{\pi^2}{6}$, show that $S_n$ diverges as $n \to \infty$. **[4 marks]**

(c) Use the method of induction to prove that

$$\sum_{r=1}^{n} r(r-1) = \frac{n(n^2-1)}{3}.$$

**[7 marks]**

**Total 25 marks**

**4.** (a) A function is defined as $g(x) = e^{3x+1}$.

(i) Obtain the Maclaurin series expansion for $g(x)$ up to and including the term in $x^4$. **[6 marks]**

(ii) Hence, estimate $g(0.2)$ correct to three decimal places. **[3 marks]**

(b) (i) Let $f(x) = x - 3 \sin x - 1$.

Use the intermediate value theorem to show that $f$ has at least one root in the interval $[-2, 0]$. **[3 marks]**

(ii) Use at least three iterations of the method of interval bisection to show that

$$f(-0.538) \approx 0 \text{ in the interval } [-0.7, -0.3].$$

($\approx 0$ means approximately equal to 0) **[8 marks]**

(c) Use the Newton–Raphson method with initial estimate $x_1 = 5.5$ to approximate the root of $g(x) = \sin 3x$ in the interval $[5, 6]$, correct to two decimal places. **[5 marks]**

**Total 25 marks**

## SECTION C

### Module 3

**Answer BOTH questions.**

**5.** (a) Ten students from across CARICOM applied for mathematics scholarships. Three of the applicants are females and the remaining seven are males. The scholarships are awarded to four successful students. Determine the number of possible ways in which a group of FOUR applicants may be selected if

(i) no restrictions are applied **[1 mark]**

(ii) at least one of the successful applicants must be female. **[3 marks]**

(b) Numbers are formed using the digits 1, 2, 3, 4 and 5 without repeating any digit. Determine

(i) the greatest possible amount of numbers that may be formed **[4 marks]**

(ii) the probability that a number formed is greater than 100. **[3 marks]**

(c) A system of equations is given as

$$\begin{aligned} 2x + 3y - z &= -3.5 \\ x - y + 2z &= 7 \\ 1.5x + 3z &= 9 \end{aligned}$$

(i) Rewrite the system of equations as an augmented matrix. **[2 marks]**

(ii) Use elementary row operations to reduce the system to echelon form. **[5 marks]**

(iii) Hence, solve the system of equations. **[3 marks]**

(iv) Show that the system has no solution if the third equation is changed to

$$1.5x - 1.5y + 3z = 9.$$ **[4 marks]**

**Total 25 marks**

**6.** (a) Alicia's chance of getting to school depends on the weather. The weather can be either rainy or sunny. If it is a rainy day, the probability that she gets to school is 0.7. In addition, she goes to school on 99% of the sunny school days. It is also known that 32% of all school days are rainy.

(i) Construct a tree diagram to show the probabilities that Alicia arrives at school. **[3 marks]**

(ii) What is the probability that Alicia is at school on any given school day? **[3 marks]**

(iii) Given that Alicia is at school today, determine the probability that it is a rainy day. **[4 marks]**

(b) (i) Show that the equation $y + xy + x^2 = 0$ is a solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - x^2}{x(1 + x)}.$$ **[5 marks]**

(ii) A differential equation is given as $y'' - 2y = 0$.

a) Find the general solution of the differential equation. **[3 marks]**

b) Hence, show that the solution which satisfies the boundary conditions

$$y\,(0) = 1 \text{ and } y'\left(\frac{\sqrt{2}}{2}\right) = 0 \text{ is}$$

$$y = \frac{1}{e^2 + 1}\left(e^{\sqrt{2}\,x} + e^{2-\sqrt{2}\,x}\right).$$ **[7 marks]**

**Total 25 marks**

**END OF TEST**

**IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.**