UNIT 1 Test 2 PREVIEW 2022

1. (a) (i) Show that $(\tan\theta - \sec\theta)^2 \equiv \frac{\sin^2\theta - 2\sin\theta + 1}{\cos^2\theta}$.

(ii) Hence, show that

$$\frac{1-\sin\theta}{1+\sin\theta} \equiv (\tan\theta - \sec\theta)^2$$

(b) Find the general solutions of the equation $\cos\theta = 2\sin^2\theta - 1$.

$$\boldsymbol{\left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\} + 2n\pi}$$

2. Determine the exact coordinates of the points of intersection between the circle with equations

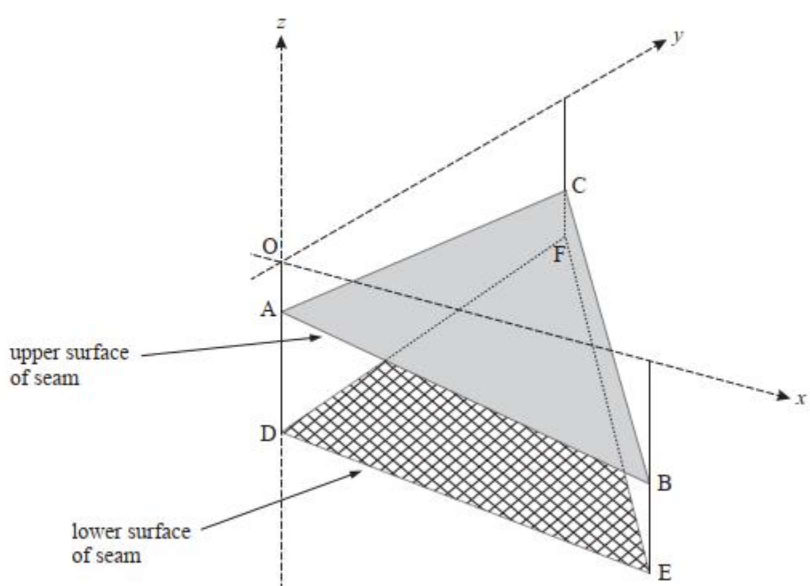
$$(x+1)^2 + (y-2)^2 = 4$$
$$(x+3)^2 + (y+1)^2 = 9$$

$$\boldsymbol{(-3, 2), \left(-\frac{3}{13}, \frac{2}{13}\right)}$$

3. A point $P(x, y)$ moves so that its distance from the fixed point (0, 3) is two times the distance from the fixed point (5, 2). Show that the equation of the locus of the point $P(x, y)$ is a circle.

Circle with centre $\left(\frac{20}{3}, \frac{5}{3}\right)$ and radius $\frac{\sqrt{104}}{3}$

4. The upper and lower surfaces of a coal seam are modelled as planes ABC and DEF, as shown below. All dimensions are metres.



Relative to axes Ox (due east), Oy (due north) and Oz (vertically upwards), the coordinates of the points are as follows.

A: $(0, 0, -15)$ B: $(100, 0, -30)$ C: $(0, 100, -25)$

D: $(0, 0, -40)$ E: $(100, 0, -50)$ F: $(0, 100, -35)$

(i) Verify that the cartesian equation of the plane ABC is $3x + 2y + 20z + 300 = 0$.

(ii) Find the vectors $\overrightarrow{DE}$ and $\overrightarrow{DF}$. Show that the vector $2i - j + 20k$ is perpendicular to each of these vectors. Hence find the cartesian equation of the plane DEF.

(iii) By calculating the angle between their normal vectors, find the angle between the planes ABC and DEF.

It is decided to drill down to the seam from a point $R(15, 34, 0)$ in a line perpendicular to the upper surface of the seam. This line meets the plane ABC at the point S.

(iv) Write down a vector equation of the line RS.
Calculate the coordinates of S.

(ii) $\boldsymbol{\overrightarrow{DE} = \begin{pmatrix} 100 \\ 0 \\ -10 \end{pmatrix}, \overrightarrow{DF} = \begin{pmatrix} 0 \\ 100 \\ 5 \end{pmatrix}, 2x - y + 20z + 800 = 0}$ **(iii)** $\boldsymbol{8.95°}$ **(iv)** $\boldsymbol{(12, 32, -20)}$

5. A curve C is given by the parametric equations

$$x = t - 1, \quad y = t^2 + 1, \quad t \in \mathbb{R}$$

Determine the Cartesian equation of C in the form $y = ax^2 + bx + c$.

$\boldsymbol{y = x^2 + 2x + 2}$

1) a) i) $(\tan\theta - \sec\theta)^2$

$= \left(\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\right)^2$

$= \left(\frac{\sin\theta - 1}{\cos\theta}\right)^2$

$= \frac{(\sin\theta - 1)^2}{\cos^2\theta}$

$= \frac{\sin^2\theta - 2\sin\theta + 1}{\cos^2\theta}$

$=$ RHS

ii) $(\tan\theta - \sec\theta)^2$

$= \frac{(\sin\theta - 1)^2}{\cos^2\theta}$

$= \frac{(\sin\theta - 1)(\sin\theta - 1)}{1 - \sin^2\theta}$

$= \frac{-(1-\sin\theta)(\sin\theta - 1)}{(1-\sin\theta)(1+\sin\theta)}$

$= \frac{-(\sin\theta - 1)}{1+\sin\theta}$

$= \frac{1-\sin\theta}{1+\sin\theta}$

b) $\cos\theta = 2\sin^2\theta - 1$

$\cos\theta = 2(1-\cos^2\theta) - 1$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ $\quad$ $\cos\theta = -1$

$\theta = \pi$

$R{\cdot}A = \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}$

$\mathrm{I}: \theta = \frac{\pi}{3}$

$\mathrm{IV}\ \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

2) $(x+1)^2 + (y-2)^2 = 4$ ①

$x^2 + 2x + 1 + y^2 - 4y + 4 = 4$

$x^2 + y^2 + 2x - 4y = -1$

$(x+3)^2 + (y+1)^2 = 9$ ②

$x^2 + 6x + 9 + y^2 + 2y + 1 = 9$

$x^2 + y^2 + 6x + 2y = -1$

$x^2 + y^2 + 2x - 4y = -1$ ③

$-\ x^2 + y^2 + 6x + 2y = -1$ ④

$-4x - 6y = 0$

$x = -\frac{3}{2}y$ ⑤

Sub ⑤ into ③

$\left(-\frac{3}{2}y\right)^2 + y^2 + 2\left(-\frac{3}{2}y\right) - 4y = -1$

$\frac{9}{4}y^2 + y^2 - 3y - 4y + 1 = 0$

$9y^2 + 4y^2 - 28y + 4 = 0$

$13y^2 - 28y + 4 = 0$

$(13y - 2)(y - 2) = 0$

$y = \frac{2}{13}$ $\quad$ $y = 2$

$x = -\frac{3}{2}\left(\frac{2}{13}\right)$ $\quad$ $x = -\frac{3}{2}(2)$

$= -\frac{3}{13}$ $\quad$ $= -3$

$\left(-\frac{3}{13}, \frac{2}{13}\right)$ $\quad$ $(-3, 2)$

General Solutions $= \frac{\pi}{3}, \pi, \frac{5\pi}{3} + 2n\pi \quad n \in \mathbb{Z}$

3) $\sqrt{(x-0)^2+(y-3)^2} = 2\sqrt{(x-5)^2+(y-2)^2}$

$x^2+(y-3)^2 = 4[(x-5)^2+(y-2)^2]$
$x^2+y^2-6y+9 = 4(x^2-10x+25+y^2-4y+4)$
$x^2+y^2-6y+9 = 4x^2-40x+100+4y^2-16y+16$
$3x^2+3y^2-40x-10y+107=0$
$x^2+y^2-\frac{40}{3}x-\frac{10}{3}y+\frac{107}{3}=0$

$x^2-\frac{40}{3}x+\frac{400}{9}+y^2-\frac{10}{3}y+\frac{25}{9} = -\frac{107}{3}+\frac{400}{9}+\frac{25}{9}$

$\left(x-\frac{20}{3}\right)^2+\left(y-\frac{5}{3}\right)^2 = \frac{104}{9}$

centre $\left(\frac{20}{3}, \frac{5}{3}\right)$ radius $= \frac{\sqrt{104}}{3}$

4) i) $3x+2y+20z+300=0$

$A(0,0,-15)$
$3(0)+2(0)+20(-15)+300=0$

At $B(100,0,-30)$
$3(100)+2(0)+20(-30)+300=0$

At $(0,100,-25)$
$3(0)+2(100)+20(-25)+300=0$
Since A, B, C satisfy the equation of the plane, $3x+2y+20z+300$ is indeed the equation of the plane.

ii) $\overrightarrow{DE} = \begin{pmatrix}100\\0\\-10\end{pmatrix}$ $\overrightarrow{DF} = \begin{pmatrix}0\\100\\5\end{pmatrix}$

$\begin{pmatrix}100\\0\\-10\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\20\end{pmatrix}$ $\begin{pmatrix}0\\100\\5\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\20\end{pmatrix}$

$=200+0-200$
$=0$

$=0-100+100$
$=0$

Since the dot products of $2i-j+20k$ and $\overrightarrow{DE}$ and $\overrightarrow{DF}$ are both equal to zero, $2i-j+20k$ is $\perp$ to both vectors and is a normal to the plane DEF.

using D

$r\cdot n = a\cdot n$
$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\20\end{pmatrix} = \begin{pmatrix}0\\0\\-40\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\20\end{pmatrix} \longrightarrow 2x-y+20z=800$

iii)

$$\cos\theta = \frac{\begin{pmatrix}3\\2\\20\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\20\end{pmatrix}}{\sqrt{3^2+2^2+20^2}\sqrt{2^2+(-1)^2+20^2}}$$

$$= \frac{404}{\sqrt{413}\sqrt{405}}$$

$$\theta = \cos^{-1}\left(\frac{404}{\sqrt{413}\sqrt{405}}\right)$$

$$= 8.95^\circ$$

iv) RS is $\perp$ to ABC $\therefore$ it is normal to ABC and parallel to $\begin{pmatrix}3\\2\\20\end{pmatrix}$

$$r = \begin{pmatrix}15\\34\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\20\end{pmatrix}$$

Solving equation of line RS and plane ABC simultaneously

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}15\\34\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\20\end{pmatrix}$$

$x = 15 + 3\lambda$ ①
$y = 34 + 2\lambda$ ②
$z = 20\lambda$ ③

Sub ①, ② and ③ into equation of plane

$3x + 2y + 20z + 300 = 0$

$3(15+3\lambda) + 2(34+2\lambda) + 20(20\lambda) + 300 = 0$

$45 + 9\lambda + 68 + 4\lambda + 400\lambda + 300 = 0$

$413\lambda = -413$

$\lambda = -1$

Sub $\lambda = -1$ into equation of line RS

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}15\\34\\0\end{pmatrix} - \begin{pmatrix}3\\2\\20\end{pmatrix} = \begin{pmatrix}12\\32\\-20\end{pmatrix}$$

coordinates of C $(12, 32, -20)$

5) $x = t - 1$

$x + 1 = t$

$y = t^2 + 1$

$y = (x+1)^2 + 1$

$= x^2 + 2x + 1 + 1$

$= x^2 + 2x + 2$