Cape Unit 1 Paper 2
2013 Solutions

1. (a)

p	q	$p \to q$	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	T	F
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(b)

$$\begin{aligned} 2 \oplus x &= (2)^2 + x^2 + 2(2) + x - 5x(2) \\ &= 8 + x^2 - 9x \\ & x^2 - 9x + 8 = 0 \\ & (x-8)(x-1) = 0 \\ & x = 8, x = 1 \end{aligned}$$

(c) For $n = 1$

$P(1) = 5 + 3 = 8$, which is a multiple of 2

Suppose the result is true for $n = k$

then $P(k) = 5^k + 3 = 2M, M \in N$

$$\begin{aligned} P(k+1) &= 5^{k+1} + 3 \\ &= 5^k[1+4] + 3 \\ &= 5^k + 3 + 4(5^k) \\ &= 2M + 4(5^k) = 2[M + 2(5^k)] \end{aligned}$$

which is a multiple of two

$\therefore$ the result is true for all positive integers

(d) (i)

$$f(-1) = -1 - 9 - p + 16 = 0$$
$$\Rightarrow p = 6$$

(ii)

$$\begin{array}{r|rrrr} -1 & 1 & -9 & 6 & 16 \\ & & -1 & 10 & -16 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$$\begin{aligned} f(x) &= (x^2 - 10x + 16)(x+1) \\ &= (x-8)(x-2)(x+1) \end{aligned}$$

OR

$$\begin{array}{r l} & \quad\quad\quad\quad \underline{x^2 - 10x + 16} \\ x+1 \big) & x^3 - 9x^2 + 6x + 16 \\ & \underline{x^3 + x^2} \\ & \quad -10x^2 + 6x \\ & \quad \underline{-10x^2 - 10x} \\ & \quad\quad\quad 16x + 16 \\ & \quad\quad\quad \underline{16x + 16} \end{array}$$

$$\begin{aligned} f(x) &= (x^2 - 10x + 16)(x+1) \\ &= (x-8)(x-2)(x+1) \end{aligned}$$

OR

$$f(x)=(x+1)(ax^2+bx+c)=x^3-9x^2+6x+16$$

$$x^3 : a=1$$

$$x^2 : a+b=-9 \to b=-10$$

$$\text{const} : c=16$$

$$f(x)=(x^2-10x+16)(x+1)$$

$$=(x-8)(x-2)(x+1)$$

(iii) $(x-8)(x-2)(x+1)=0$

$x=8, x=2, x=-1$

2. (a) $f(x)=x^2-x$

A function is 1 – 1 if and only if $f(x_1)=f(x_2)\to x_1=x_2$

Let $x_1, x_2 \in A$ and $x_1 \neq x_2$

Then $f(x_1)=f(x_2)$

i.e. $x_1{}^2-x_1=x_2{}^2-x_2$

$(x_1-x_2)(x_1+x_2)=x_1-x_2$

$x_1+x_2=1$

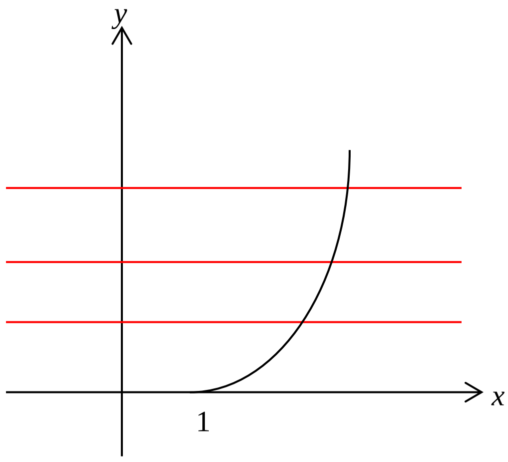
$x_2=1-x_1$

But $x_1 \in A \to x_1 \geq 1\ x_1$, hence $x_2 \leq 0$

Hence $x_2 \notin A$, which is a contradiction to the assumption $x_2 \in A$.

$\therefore x_1=x_2 \to f$ is 1 – 1.

OR



(b) (i) (a) $f(x)=3x+2$

$y=3x+2$

$x=3y+2$

$\frac{x-2}{3}=y \to f^{-1}(x)=\frac{x-2}{3}$

$g(x)=e^{2x}$

$y=e^{2x}$

$x=e^{2y}$

$\ln x=2y$

$\frac{\ln x}{2}=y \to g^{-1}=\frac{\ln x}{2}$

(b) $f \circ g(x)=3e^{2x}+2$

(ii) $y = 3e^{2x} + 2$

$$\frac{y-2}{3} = 3e^{2x}$$

$$\frac{x-2}{3} = 3e^{2y}$$

$$\frac{1}{2}\ln\left(\frac{x-2}{3}\right) = y \rightarrow (f \circ g)^{-1}(x) = \frac{1}{2}\ln\left(\frac{x-2}{3}\right)$$

$$g^{-1}(x) \circ f^{-1}(x) = g^{-1}\left(\frac{x-2}{3}\right)$$

$$= \frac{1}{2}\ln\left(\frac{x-2}{3}\right)$$

$$\therefore (f \circ g)^{-1}(x) = g^{-1}(x) \circ f^{-1}(x)$$

(c) (i) $3x^2 + 4x - 4 \le 0$

$(x+2)(3x-2) \le 0$

<u>case I</u>

$x + 2 \ge 0$ and $3x - 2 \le 0$

$x \ge -2$ and $x \le \frac{2}{3}$

$-2 \le x \le \frac{2}{3}$

<u>case II</u>

$x + 2 \le 0$ and $3x - 2 \ge 0$

$x \le -2$ and $x \ge \frac{2}{3}$ (not possible)

$\therefore -2 \le x \le \frac{2}{3}$

(ii) $|x+2| = 3x + 5$

$(x+2)^2 = (3x+5)^2$

$8x^2 + 26x + 21 = 0$

$$x = \frac{-26 \pm \sqrt{4}}{16}$$

$$x = \frac{-24}{16} = -\frac{3}{2}$$

$x = \frac{-28}{16} = -\frac{7}{4}$ (inadmissible)

OR

$x + 2 = 3x + 5$

$x = -\frac{3}{2}$

$x + 2 = -(3x+5)$

$x = -\frac{7}{4}$ (inadmissible)

3. (a) (i) $$\text{R.H.S.} = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\frac{\sin\theta}{\cos\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta}$$

$$= 2\sin\theta\cos\theta$$

$$= \sin 2\theta$$

(ii) $\sin 2\theta - \tan\theta = 0$

$$\frac{2\tan\theta}{1+\tan^2\theta} - \tan\theta = 0$$

$$2\tan\theta - \left(1+\tan^2\theta\right)\tan\theta = 0$$

$$\tan\theta - \tan^3\theta = 0$$

$$\tan\theta\left(1-\tan^2\theta\right) = 0$$

$$\tan\theta = 0$$

$$\theta = 0, \pi, 2\pi$$

$$(0, 3.14, 6.28)$$

$$\tan\theta = \pm 1$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$(0.785, 3.93, 2.36, 5.5)$$

NB Solutions in degrees will be accepted.

(b) (i) $f(\theta) = 3\cos\theta - 4\sin\theta$

$$r = \sqrt{4^2+3^2} = 5$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

$$= 0.927\,\text{rad}$$

$$\therefore\ f(\theta) = 5\cos(\theta + 0.927)$$

(ii) (a) max $\cos(\theta+\alpha) \rightarrow$ max $f(\theta)$

$$r = 5$$

(b) min value of $f(\theta) \rightarrow$ min of $\dfrac{1}{8+f(\theta)}$

min value $\dfrac{1}{8+5} = \dfrac{1}{13}$

(iii) (a) $A+B+C=\pi \rightarrow A = \pi - (B+C)$

$$\rightarrow \sin A = \sin[\pi - (B+C)]$$

$$\rightarrow \sin A = \sin(B+C)$$

(b) from (a) $\sin B = \sin(C+A)$, $\sin C = \sin(A+B)$

$$\rightarrow \sin A + \sin B + \sin C$$

$$= \sin(B+C) + \sin(C+A) + \sin(A+B)$$

4. (a) (i) $(x-3)^2 + (y-2)^2 = 9$

centre $(3,2)$

radius $r = 3$

(ii) (a) in a circle tangent is perpendicular to normal.

gradient of normal $= \dfrac{2-2}{6-3} = 0$

$\therefore$ eqn. of tangent is $y = 2$

since it passes through $(3,2)$

(b) eqn. of tangent is $x = k$

eqn. of tangent is $x = 6$

hence the tangent is parallel to the $y-$axis.

(b) $y = 2t - 4 \rightarrow t = \frac{y+4}{2}$

$$x = \left(\frac{y+4}{2}\right)^2 + \left(\frac{y+4}{2}\right)$$

$$= \frac{y^2 + 8y + 16}{4} + \frac{y+4}{2}$$

$$= \frac{y^2 + 8y + 16 + 2y + 8}{4} \rightarrow 4x = y^2 + 8y + 16 + 2y + 8$$

(c) (i) $\overrightarrow{AB} =$ (**i** +2**j** – 4**k**) – (3**i** – **j** + 2**k**)
= -2**i** + 3**j** – 6**k**

$\overrightarrow{BC} =$ (-**i** + **j** – 2**k**) **– (i** + 2**j** – 4**k**)
= -2**i** – **j** + 2**k**

(ii) **r** is perpendicular to the plane through A , B and C
if **r**. $\overrightarrow{AB} = 0$ and **r**. $\overrightarrow{BC} = 0$

r. $\overrightarrow{AB} =$ (-16**j** – 8**k**).(-2**i** + 3**j** – 6**k**)
= **-** 48 + 48 = 0

$\overrightarrow{BC}$.**r** = (-2**i** – **j** + 2**k**)(-16**j** – 8**k**)
= 16 – 16 =0

OR

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{bmatrix} i & j & k \\ -2 & 3 & -6 \\ -2 & -1 & 2 \end{bmatrix}$$

$$= i(6-6) - j(-4-12) + k(2+6)$$

$$= 16j + 8k = (-1)[-16j - 8k]$$

$\therefore -16j - 8k$ is perpendicular to the plane through A, B, C .

(iii) let **r** = x**i** + y**j** + z**k** with **r**.**n** = **d**

at the point A, **r** = 3**i** – **j** + 2**k** and **n** = -16**j** – 8**k**
$\therefore d =$ (3**i** – **j** + 2**k**).(-16**j** – 8**k**) = 16 – 16 = 0

Hence Cartesian equation of the plane is
(x**i** + y**j** + z**k**).(-16**j** – 8**k**) = 0
$-16y - 8z = 0 \rightarrow 2y + z = 0$

5. (a) (i) $\lim_{x \to 2+} f(x) = 4$
$\lim_{x \to 2-} f(x) = 4$
$\therefore \lim_{x \to 2} f(x) = 4$

(ii) $f(x)$ is not defined at $x = 2$
$\therefore$ $f(x)$ is not continuous

(b) $y = \frac{x^2 + 2x + 3}{(x^2 + 2)^3}$

$u = x^2 + 2x + 3; \frac{du}{dx} = 2x + 2$

$v = \left(x^2 + 2\right)^3; \frac{dv}{dx} = 3\left(x^2 + 2\right)^2 (2x) = 6x\left(x^2 + 2\right)^2$

$\frac{dy}{dx} = \frac{\left(x^2 + 2\right)^3 (2x + 2) - \left(x^2 + 2x + 3\right)6x\left(x^2 + 2\right)^2}{\left(x^2 + 2\right)^4}$

$\frac{dy}{dx} = \frac{\left(x^2 + 2\right)^2 \left[\left(x^2 + 2\right)(2x + 2) - \left(x^2 + 2x + 3\right)6x\right]}{\left(x^2 + 2\right)^6}$

$\frac{dy}{dx} = \frac{-4x^3 - 10x^2 - 14x + 4}{\left(x^2 + 2\right)^4}$

(c) $x = 1 - 3\cos\theta; \frac{dx}{d\theta} = 3\sin\theta$

$y = 2\sin\theta; \frac{dy}{d\theta} = 2\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$= 2\cos\theta \times \frac{1}{3\sin\theta}$

$= \frac{2}{3}\cot\theta$

(d) (i) $y = x^2 + 3, y = 4x$

$x^2 + 3 = 4x$

$x^2 - 4x + 3 = 0 \rightarrow (x - 3)(x - 1) = 0$

$x = 3, x = 1$

$P(1,4)$ and $Q(3,12)$

(ii) Area of shaded region

$= \int_1^3 \left(4x - x^2 - 3\right)dx$

$= \left[4\frac{x^2}{2} - \frac{x^3}{3} - 3x\right]_1^3$

$= \left(\frac{36}{2} - \frac{27}{3} + 9\right) - \left(\frac{4}{2} - \frac{1}{3} - 3\right)$

$(18 - 9 - 9) - \left(2 - \frac{1}{3} - 3\right) = 1\frac{1}{3}$

6. (a) (i) $\int x(1 - x)^2 dx$

$u = 1 - x \rightarrow du = -dx$

$\rightarrow x = 1 - u$

we have $-\int (1 - u)u^2 du = \int \left(u^3 - u^2\right)du$

$= \frac{1}{4}u^4 - \frac{1}{3}u^3 + c$

$= \frac{1}{4}(1 - x)^4 - \frac{1}{3}(1 - x)^3 + c$

(ii) $\int [f(t) + g(t)]dt = \int [5\cos t + 4\sin t]dt$

$= 5\sin t - \frac{4\cos 5t}{5} + c$

$\int f(t)dt = \int 2\cos t dt = 2\sin t + c_1$

$\int g(t)dt = \int (4\sin 5t + 3\cos t)dt = \frac{-4\cos 5t}{5} + 3\sin t + c_2$

$$\therefore \int f(t)dt + \int g(t)dt = 5\sin t - \frac{4}{5}\cos 5t + c = \int \left[f(t) + g(t) \right] dt$$

(b) (i) $p = 2x + 2r + \pi r = 2x + r(2+\pi)$

$$600 = 2x + r(2+\pi)$$

$$r = \frac{600 - 2x}{2+\pi}$$

(ii) $A = 2rx + \frac{1}{2}\pi r^2$

$$= 2x\left(\frac{600-2x}{2+\pi}\right) + \frac{1}{2}\pi\left(\frac{600-2x}{2+\pi}\right)^2$$

$$= \frac{1200x - 4x^2}{2+\pi} - \frac{\pi}{2(2+\pi)^2}(600-2x)^2$$

$$\frac{dA}{dx} = \frac{1200x - 8x}{2+\pi} - \frac{2\pi}{(2+\pi)^2}(600-2x)$$

$$\frac{d^2A}{dx^2} = \frac{-8}{2+\pi} + \frac{4\pi}{(2+\pi)^2} < 0 \rightarrow \max$$

$$\frac{dA}{dx} = 0 \rightarrow (2+\pi)(1200-8x) - 2\pi(600-2x) = 0$$

$$2400 - 16x - 4x\pi = 0$$

$$x = \frac{2400}{16+4\pi} \approx 84\,\text{m}$$

(c) (i) $y' = -x\cos x - \sin x + 2\sin x + A$

$$= -x\cos x + \sin x$$

$$y'' = x\sin x - \cos x + \cos x$$

$$= x\sin x$$

(ii) At $x = 0$

$$1 = -2 + B$$
$$B = 3$$

At $x = \pi$

$$6 = 2 + \pi A + 3 \rightarrow A = \frac{1}{\pi}$$

i.e. $y = -x\sin x - 2\cos x + \frac{x}{11} + 3$