Solutions to SBA Unit 2 Test 2 (2016)

Question 1

(a) $-1 < \log_2 y < 1$ (1)

$\frac{1}{2} < y < 2$ (1) [2]

(b) $S_\infty = \frac{a}{1-r}$

$\frac{\log_2 27}{1 - \log_2 y} = 3$ (1)

$\log_2 27 = 3 - 3\log_2 y$ (1)

$\log_2 27 + 3\log_2 y = 3$ (1)

$\log_2 27y^3 = 3$ (1)

$27y^3 = 2^3$ (1)

$y^3 = \frac{8}{27}$ (1)

$y = \frac{2}{3}$ (1) [7]

Question 2

(a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ (1) [1]

(b) $\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$ [2]

(1) (1)

(c) $x = \frac{1}{2}$ (1)

$\ln\left(1 + \left(\frac{1}{2}\right)^2\right) = \left(\frac{1}{2}\right)^2 - \frac{\left(\frac{1}{2}\right)^4}{2} + \frac{\left(\frac{1}{2}\right)^6}{3} - \frac{\left(\frac{1}{2}\right)^8}{4} +$ (1)

$\ln\frac{5}{4} = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$ (1)

$\ln\frac{5}{4} = \frac{1}{4}\left[1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^4 - \frac{1}{4}\left(\frac{1}{2}\right)^6 + \cdots\right]$ (1)

$4\ln\frac{5}{4} = 1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^4 - \frac{1}{4}\left(\frac{1}{2}\right)^6 + \cdots$ (1) [5]

Question 3

(a) $\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$ (1)

$1 = A(r+2) + Br$ (1)

Let $r = -2$

$1 = -2B$

$B = -\frac{1}{2}$ (1)

Let $r = 0$

$1 = 2A$

$A = \frac{1}{2}$ (1)

$\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$ (1) [5]

(b) $$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \left[\begin{array}{c} \frac{1}{2} - \frac{1}{6} \\ + \frac{1}{4} - \frac{1}{8} \\ + \frac{1}{6} - \frac{1}{10} \\ + \frac{1}{8} - \frac{1}{12} \\ \vdots \\ + \frac{1}{2(n-1)} - \frac{1}{2(n+1)} \\ + \frac{1}{2n} - \frac{1}{2(n+2)} \end{array}\right]$$ (1) (1)

$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$ (1)

$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$ (1) (1)

$= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)}$ (1)

$= \frac{3n^2 + 5n}{4(n+1)(n+2)}$ (1)

$= \frac{n(3n+5)}{4(n+1)(n+2)}$ (1) [8]

(c) $\sum_{r=n+1}^{2n}\frac{1}{r(r+2)} = S_{2n} - S_n$

$$= \frac{2n(6n+5)}{4(2n+1)(2n+2)} - \frac{n(3n+5)}{4(n+1)(n+2)} \quad (1)$$

$$= \frac{n(6n+5)}{4(2n+1)(n+1)} - \frac{n(3n+5)}{4(n+1)(n+2)}$$

$$= \frac{n(6n+5)(n+2) - n(3n+5)(2n+1)}{4(n+1)(n+2)(2n+1)} \quad (1)\ (1)$$

$$= \frac{n[6n^2+17n+10-6n^2-13n-5]}{4(n+1)(n+2)(2n+1)} \quad (1)$$

$$= \frac{n[4n+5]}{4(n+1)(n+2)(2n+1)} \ \textit{as required} \quad (1)[5]$$

Question 4

(a) $\sqrt{1-8x} = (1-8x)^{\frac{1}{2}}$

$$= 1 + \frac{1}{2}(-8x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-8x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-8x)^3$$

$$= 1 - 4x - 8x^2 - 32x^3$$

(1) (1) (1) (1) [4]

(b) Valid for $-1 < -8x < 1$

$$-\frac{1}{8} < x < \frac{1}{8} \quad (1) \quad [1]$$

(c) $x = \frac{1}{100}$

$$\left(1 - 8\left(\frac{1}{100}\right)\right)^{\frac{1}{2}} (1) = \left(\frac{23}{25}\right)^{\frac{1}{2}} = \frac{\sqrt{23}}{5} (1)$$

$$\frac{\sqrt{23}}{5} = 1 - 4\left(\frac{1}{100}\right) - 8\left(\frac{1}{100}\right)^2 - 32\left(\frac{1}{100}\right)^3 \quad (1)$$

$$= 0.959168$$

$$\sqrt{23} = 5(0.959168) \quad (1)$$

$$= 4.79584 \quad (1) \quad [5]$$

Question 5

(a) $y = 2^x$ (1)

$$\ln y = \ln 2^x$$

$$\ln y = x \ln 2 \quad (1)$$

$$\frac{1}{y}\frac{dy}{dx} = \ln 2 \quad (1)$$

$$\frac{dy}{dx} = y \ln 2$$

$$= 2^x \ln 2 \ \textit{as required} \quad (1) \quad [4]$$

(b) i $f(1) = 2^1 + 1 - 4 = -1$

$f(2) = 2^2 + 2 - 4 = 2$ (1)

The function is **continuous** (1) and there is a **sign change** (1) so by the **IVT** (1) there is a root α in the interval [1,2]. [4]

ii. $a = 1 \qquad f(a) = -1$

$b = 2 \qquad f(b) = 2$

$$\alpha = \frac{af(b) - bf(a)}{f(b) - f(a)} \quad (1)$$

$$= \frac{1(2) - 2(-1)}{2 - (-1)} \quad (1)$$

$$= \frac{4}{3} \quad (1) \quad [3]$$

iii. $f'(x) = 2^x \ln 2 + 1$ (1)

$$x_2 = 1 - \frac{2^1 + 1 - 4}{2^1 \ln 2 + 1} \quad (1)\ (1)$$

$$= 1.419 \quad (1) \quad [4]$$