Solution to SBA Unit 2 Test 1

1. a) $y^3 + 3y = x^3$

$3y^2 \frac{dy}{dx} + 3\frac{dy}{dx} = 3x^2$ (1) + (1) + (1)

$(3y^2 + 3)\frac{dy}{dx} = 3x^2$ (1)

$\frac{dy}{dx} = \frac{3x^2}{3y^2 + 3}$

$\frac{dy}{dx} = \frac{x^2}{y^2 + 1}$ as required (1) [5]

(b) $f_x(x, y) = ye^{xy}$ (1)

$f_{xy} = xye^{xy} + e^{xy}$ (1) + (1) [3]

(c) i) $\frac{dx}{dt} = 8\sin t \cos t$ (1)

$\frac{dy}{dt} = -2\sin t$ (1)

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= -\frac{2\sin t}{8\sin t \cos t}$ (1)

$= -\frac{1}{4\cos t}$

$= -\frac{1}{4}\sec t$ (1) [4]

ii) $1 = 2\cos t$

$\frac{1}{2} = \cos t$ (1)

$\frac{\pi}{3} = t$ (1) [2]

OR

$3 = 4\sin^2 t$

$\frac{3}{4} = \sin^2 t$ (1)

$\frac{\sqrt{3}}{2} = \sin t$

$t = \frac{\pi}{3}$ (1)

iii) $\frac{dy}{dx} = -\frac{1}{4\cos\frac{\pi}{3}} = -\frac{1}{2}$ (1)

gradient of normal = 2 (1)

$y = mx + c$

$1 = 2(3) + c$

$-5 = c$ (1)

$y = 2x - 5$ as required [3]

2. a)i) $\frac{4x}{(3x+1)(x+1)^2} = \frac{A}{3x+1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$4x = A(x+1)^2 + B(x+1)(3x+1) + C(3x+1)$ (1)

let $x = -1$

$-4 = -2C$

$2 = C$ (1)

$let\ x = -\frac{1}{3}$

$-\frac{4}{3} = \frac{4}{9}A$

$-3 = A$ (1)

Compare coefficients x^2

$0 = A + 3B$

$0 = -3 + 3B$

$1 = B$ (1)

$\frac{4x}{(3x+1)(x+1)^2} = -\frac{3}{3x+1} + \frac{1}{x+1} + \frac{2}{(x+1)^2}$ (1) [5]

ii) $\int_0^1 \frac{4x}{(3x+1)(x+1)^2}\,dx$

$= \int_0^1 -\frac{3}{3x+1} + \frac{1}{x+1} + \frac{2}{(x+1)^2}\,dx$

$= \left[-\ln(3x+1) + \ln(x+1) - \frac{2}{x+1}\right]_0^1$ (1) + (1) + (1)

$= -\ln 4 + \ln 2 - 1 - (-\ln 1 + \ln 1 - 2)$ (1)

$= -2\ln 2 + \ln 2 + 1$

$= 1 - \ln 2$ as required (1) [5]

(b) i) $x = e^u$

$\frac{dx}{du} = e^u$

$dx = e^u du$ (1)

$\int \frac{2 + \ln x}{x^2} \, dx$

$= \int \frac{2 + \ln e^u}{(e^u)^2} \, e^u du$ (1)

$= \int \frac{2 + u}{e^u} \, du$

$= \int (2 + u)e^{-u} \, du$ as required (1) [3]

ii) $x = 1$ $u = 0$ (1)

$x = e$ $u = 1$ (1)

$\int_1^e \frac{2 + \ln x}{x^2} \, dx$

$= \int_0^1 (2 + u)e^{-u} \, du$

$u = 2 + u$ $\frac{du}{dx} = 1$ (1)

$\frac{dv}{dx} = e^{-u}$ $v = -e^{-u}$ (1)

$\int_0^1 (2 + u)e^{-u} \, du$

$= -[(2 + u)e^{-u}]_0^1 + \int_0^1 e^{-u} \, du$ (1)

$= -[(2 + u)e^{-u}]_0^1 - [e^{-u}]_0^1$ (1)

$= -[(2 + 1)e^{-1} - (2 + 0)e^{-0}] - [e^{-1} - e^{-0}]$

$= -3e^{-1} + 2 - e^{-1} + 1$

$= -4e^{-1} + 3$ (1) [7]

(c) $h = \frac{3-1}{4} = \frac{1}{2}$ (1)

x	1	1.5	2	2.5	3
y	0.25	0.15686	0.09091	0.05369	0.03333

$\int_1^3 \frac{1}{x^3 + 3} \, dx$

$= \frac{1}{2}\left(\frac{1}{2}\right)[0.25 + 0.03333$

$+ 2(0.15686 + 0.09091 + 0.05369)]$ (1)

$= 0.222$ (1) [3]

3. (a) i) $u = 1 + i\sqrt{3}$

$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$ (1)

$\theta = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$ (1)

$u = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (1) [3]

ii) $u^2 = 2^2\left(\cos\left(2 \times \frac{\pi}{3}\right) + i\sin\left(2 \times \frac{\pi}{3}\right)\right)$ (1)

$= 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ (1) [2]

iii) $z^2 - 2z + 4 = 0$

let $z = u$

$u^2 - 2u + 4$

$= 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) - 2\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]$

$+ 4$ (1)

$= 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) - 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) + 4$

$= -2 + 2\sqrt{3} - 2 - 2\sqrt{3} + 4$

$= 0$

Therefore u is a root of $z^2 - 2z + 4 = 0$. (1)

OR

$z^2 - 2z + 4 = 0$

$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$ (1)

$$z = \frac{2 \pm \sqrt{-12}}{2}$$

$$z = \frac{2 \pm 2\sqrt{3}i}{2}$$

$$z = -\sqrt{3}\, i + 1 \text{ or } z = 1 + \sqrt{3}\, i$$

Therefore u is a root of $z^2 - 2z + 4 = 0$. (1) [2]

iv) $Other\ root\ u^* = 1 - i\sqrt{3}$ (1) [1]

(b) i) $|z + 1 + 2i| = \sqrt{2}|z - 1|$

$|x + iy + 1 + 2i| = \sqrt{2}|x + iy - 1|$ (1)

$|(x + 1) + (y + 2)i| = \sqrt{2}|(x - 1) + iy|$

$(x + 1)^2 + (y + 2)^2 = 2[(x - 1)^2 + y^2]$ (1)

$x^2 + 2x + 1 + y^2 + 4y + 4 = 2x^2 - 4x + 2 + 2y^2$

$x^2 - 6x + y^2 - 4y - 3 = 0$ (1)

$(x - 3)^2 - 9 + (y - 2)^2 - 4 = 3$ (1)

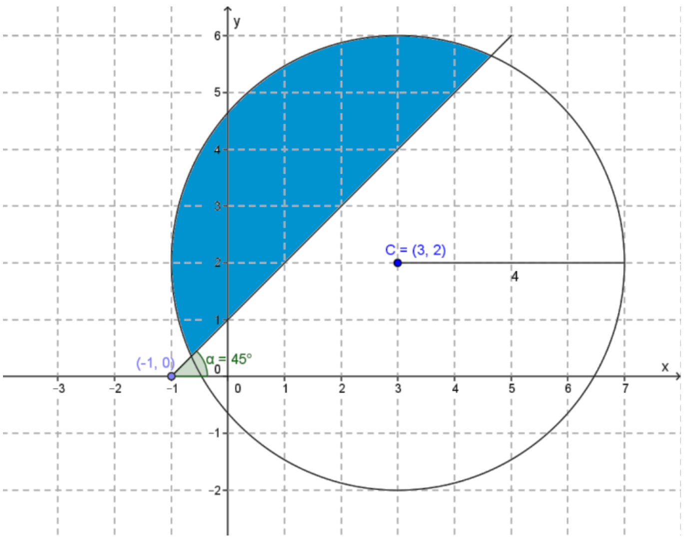
$(x - 3)^2 + (y - 2)^2 = 16$ (1)

Therefore P is a circle with $C(3, 2)$ (1) and $r = 4$ (1) [7]

ii) $\tan[\arg(z + 1)] = 1$

$\arg(z + 1) = \tan^{-1} 1$

$\arg(z + 1) = \frac{\pi}{4}$



Circle with Centre $(3, 2)$ (1) Radius $= 4$ (1)

Line with angle $45°$ (1) Starting point $(-1, 0)$ (1)

[4]

iii) shading inside the circle and above the line (1)

[1]