HARRISON COLLEGE INTERNAL EXAMINATION 2021

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION

SCHOOL BASED ASSESSMENT

PURE MATHEMATICS

UNIT 2 – TEST 2

Time: 1 hour and 20 minutes

NAME OF STUDENT: ____

SCHOOL CODE: 030014

DATE: ________________________________

This examination paper consists of 10 printed pages, including 1 blank page.

The paper consists of 5 questions.

The maximum mark for this examination is 60.

INSTRUCTIONS TO CANDIDATES

1. Write your name clearly in the space above.
2. Answer **ALL** questions in the **SPACES PROVIDED**.
3. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided. You must also write your name and candidate number clearly on any additional paper used.
4. Number your questions **carefully and identically to those on the question paper**.
5. Unless otherwise stated in the question, any numerical answer that is not exact, **MUST** be written correct to three (3) significant figures.

EXAMINATION MATERIALS ALLOWED

1. Mathematical formulae
2. Electronic calculator (non-programmable, non-graphical).

1. a) In the expansion of $(3+px)^6$, the coefficient of x^4 is four times the coefficient of the x^2 term. Find the possible values of p.

$$(3+px)^6 = 3^6 + {}^6C_1 3^5(px)^1 + {}^6C_2 3^4(px)^2 + {}^6C_3 3^3(px)^3 + {}^6C_4 3^2(px)^4 + {}^6C_5 3^1(px)^5 + (px)^6$$

$$coefficient\ of\ x^4 = {}^6C_4 3^2(p)^4 = 15\times 9p^4$$ 1

$$coefficient\ of\ x^2 = {}^6C_2 3^4(p)^2 = 15\times 81p^2$$ 1

$$15\times 9\times p^4 = 4(15\times 81\times p^2)$$ 1

$$p^2 = \frac{4\times 15\times 81}{15\times 9} = 36 \quad therfore\ p = -6\ \ OR\ \ p = 6$$

1 1

Total 5 marks

[5 marks]

b) i) Expand $\sqrt[3]{(1+2x)}$ in ascending powers of x up to and including the term in x^3 and state the values of x for which the expansion is valid.

$$\sqrt[3]{(1+2x)} = (1+2x)^{\frac{1}{3}}$$

$$= 1 + \frac{1}{3}(2x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(2x)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(2x)^3 + \cdots$$

$$= 1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{40}{81}x^3 + \cdots$$

4 (1 mark for each of the four terms)

$$Expansion\ valid\ for -1 < 2x < 1$$

$$-\frac{1}{2} < x < \frac{1}{2}$$ 1

Total 5 marks

[5 marks]

ii) Use your expansion from i) to find an approximation for $\sqrt[3]{1.1}$ to 5 decimal places.

$$Let\ \sqrt[3]{1.1} = \sqrt[3]{1 + 2x}$$

$$1.1 = 1 + 2x, \quad x = \frac{1.1 - 1}{2} = 0.05$$ 1

1 for substitution

$$\sqrt[3]{1.1} \approx 1 + \frac{2}{3}(0.05) - \frac{4}{9}(0.05)^2 + \frac{40}{81}(0.05)^3 = 1.03228\ to\ 5\ decimal\ places$$

1

3 marks

[3 marks]

Total 13 marks

2. Find the Maclaurin series expansion for $f(x) = \sqrt{8 + e^x}, x \in \mathbb{R}$, in ascending powers of x, up to and including the term in x^2. Express each coefficient in its simplest form.

$$f(x) = \sqrt{8 + e^x} = (8 + e^x)^{\frac{1}{2}} \qquad f(0) = (8 + 1)^{\frac{1}{2}} = 3$$

$$f'(x) = \frac{1}{2}(8 + e^x)^{-\frac{1}{2}}e^x$$ 1

$$f'(0) = \frac{1}{2}(9)^{-\frac{1}{2}} = \frac{1}{6}$$ 1

$$f''(x) = \frac{1}{2}e^x\left(-\frac{1}{2}\right)(8 + e^x)^{-\frac{3}{2}}e^x + \frac{1}{2}e^x(8 + e^x)^{-\frac{1}{2}}$$

$$f''(0) = -\frac{1}{4} \times \frac{1}{27} + \frac{1}{2} \times \frac{1}{3} = \frac{17}{108}$$ 1

$$= -\frac{1}{4}e^{2x}(8 + e^x)^{-\frac{3}{2}} + \frac{1}{2}e^x(8 + e^x)^{-\frac{1}{2}}$$

1 1

$$f(x) = 3 + \frac{1}{6}x + \frac{17}{108} \times \frac{x^2}{2!} + \cdots$$

$$f(x) = 3 + \frac{1}{6}x + \frac{17}{216}x^2 + \cdots$$ 3 (1 for each term)

Total 8 marks

[Total 8 marks]

3. A sequence is defined by

$$u_1 = 2 \quad and \quad u_{n+1} = \frac{u_n}{1+u_n}$$

i) Calculate u_3.

$$u_1 = 2, \qquad u_{n+1} = \frac{u_n}{1+u_n}$$

$$u_2 = \frac{u_1}{1+u_1} = \frac{2}{1+2} = \frac{2}{3}$$ 1

$$u_3 = \frac{u_2}{1+u_2} = \frac{\frac{2}{3}}{1+\frac{2}{3}}$$

$$= \frac{2}{3} \div \frac{5}{3} = \frac{2}{5}$$ 1

Total 2 marks

[2 marks]

ii) Prove by mathematical induction that, for $n \geq 1$,

$$u_n = \frac{2}{2n-1}$$

$$Let\ P_n\ be\ u_n = \frac{2}{2n-1} \quad for\ n \geq 1$$

$$P_1 \Longrightarrow u_1 = \frac{2}{2(1)-1} = 2, \quad so\ P_1\ is\ true$$ 1

$$Assume\ that\ P_k\ is\ true\ i.e\ u_k = \frac{2}{2k-1}$$ 1

$$Show\ that\ P_{k+1}\ is\ true\ if\ P_k\ is\ true, \qquad P_{k+1} \Longrightarrow u_{k+1} = \frac{2}{2(k+1)-1} = \frac{2}{2k+1}$$ 1

$$LHS = u_{k+1} = \frac{u_k}{1+u_k}\ from\ recurrence\ formula$$ 1

$$= \frac{2}{2k-1} \div \left(1 + \frac{2}{2k-1}\right)$$ 1

$$= \frac{2}{2k-1} \div \left(\frac{2k-1+2}{2k-1}\right)$$

$$= \frac{2}{2k-1} \times \frac{2k-1}{2k+1}$$ 1 working implied or otherwise

1 $$= \frac{2}{2k+1} = RHS$$

Hence, by induction, P_n is true for all $n \geq 1$ 1

Total 8 marks

[8 marks]

Total 10 marks

4. i) Show that

$$\frac{1}{r} - \frac{1}{r+2} \equiv \frac{2}{r(r+2)}$$

$$LHS = \frac{1}{r} - \frac{1}{r+2} = \frac{1(r+2) - 1(r)}{r(r+2)}$$ 1

$$= \frac{r+2-r}{r(r+2)} = \frac{2}{r(r+2)} = RHS$$

1

Total 2 marks

[2 marks]

ii) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{2}{r(r+2)}$$

$$\sum_{r=1}^{n} \frac{2}{r(r+2)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+2}\right)$$

$$= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} +$$ 1

$$\frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} +$$

1

$$\frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{n-3} - \frac{1}{n-1} +$$

$$\frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}$$ 1

$$= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

1 1 1

Total 6 marks

[6 marks]

iii) Find the value of N, given that

$$\sum_{r=N+1}^{\infty} \frac{2}{r(r+2)} = \frac{11}{30}$$

$$\sum_{r=N+1}^{\infty} \left(\frac{1}{r} - \frac{1}{r+2}\right) = \sum_{r=1}^{\infty} \left(\frac{1}{r} - \frac{1}{r+2}\right) - \sum_{r=1}^{N} \left(\frac{1}{r} - \frac{1}{r+2}\right)$$ 1

$$= \left[\frac{3}{2} - \frac{1}{\infty} - \frac{1}{\infty}\right] - \left[\frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right]$$ 1

$$= \frac{1}{N+1} + \frac{1}{N+2}$$

$$\frac{1}{N+1} + \frac{1}{N+2} = \frac{11}{30}$$ 1

$$\frac{(N+2)+(N+1)}{(N+1)(N+2)} = \frac{11}{30}$$

$$30[N+2+N+1] = 11(N+1)(N+2)$$ 1

$$30(2N+3) = 11(N^2+3N+2)$$

$$0 = 11N^2 + 33N - 60N + 22 - 90$$

$$11N^2 - 27N - 68 = 0$$ 1

$$(11N+17)(N-4) = 0$$

$$N = -\frac{17}{11} \; OR \; N = 4$$

We take $N = 4$ 1

Total 6 marks

[6 marks]

Total 14 marks

5. a) i) Show that the equation

$$x^3 + x - 7 = 0$$

has a root between 1.6 and 1.8.

$$f(x) = x^3 + x = 7$$

$$f(1.6) = 1.6^3 + 1.6 - 7 = -1.304$$ 1

$$f(1.8) = 1.8^3 + 1.8 - 7 = 0.632$$ 1

Since $f(x)$ is continuous in the interval $1.6 \leq x \leq 1.8$ and $f(1.6) \times f(1.8) < 0$, then $x^3 + x - 7 = 0$ has at least 1 root between 1.6 and 1.8. 1

Total 3 marks

[3 marks]

ii) Use linear interpolation once, starting with the interval in a) i), to give this root two decimal places.

$$x_1 = \frac{1.6|f(1.8)| + 1.8|f(1.6)|}{|f(1.8)| + |f(1.6)|}$$

1 for formula

$$= \frac{1.6(0.632) + 1.8(1.304)}{0.632 + 1.304} = 1.74 \textit{ to 2 decimal places}$$

1

Total 2 marks

[2 marks]

b) It is known that the function

$$f(x) = 3\sqrt{x} + \frac{18}{\sqrt{x}} - 20$$

has a root α in the interval $[1.1, \; 1.2]$.

i) Find $f'(x)$.

$$f(x) = 3x^{\frac{1}{2}} + 18x^{-\frac{1}{2}} - 20$$

$$f'(x) = 3\left(\frac{1}{2}\right)x^{-\frac{1}{2}} - 18\left(\frac{1}{2}\right)x^{-\frac{3}{2}}$$

$$= \frac{3}{2}x^{-\frac{1}{2}} - 9x^{-\frac{3}{2}}$$

1 1

2 marks

[2 marks]

ii) Using $x_0 = 1.1$ as a first approximation to α, apply the Newton-Raphson procedure twice to $f(x)$ to find a third approximation to α, giving your answer to 3 significant figures.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ 1 for correct use of formula

$$x_0 = 1.1, \quad x_1 = 1.1 - \frac{f(1.1)}{f'(1.1)}$$ 1

$$= 1.1 - \frac{0.308753}{-6.370865}$$ 1

$$= 1.148$$ 1

$$x_2 = 1.148 - \frac{f(1.148)}{f'(1.148)}$$ 1

$$= 1.148 - \frac{0.014044}{-5.916968}$$ 1

$$= 1.150 = 1.15 \; to \; 3 \; s.f.$$ 1

7 marks

[7 marks]

Total 15 marks

END OF TEST