Solution to Unit 2 Test 1 (2015)

1. (a) $M = 900 - \frac{1500}{3(3)+2} = 764\ tonnes$ (1)

$$M = 900 - \frac{1500}{2 + 5\ln(3+1)}$$

$= 732$ tonnes (1) [2]

(b) $M = 900 - \frac{1500}{3t+2}$

$$M = 900 - 1500(3t+2)^{-1}$$

$$\frac{dM}{dt} = 1500(3t+2)^{-2}(1) \times 3(1)$$

$$= \frac{4500}{(3t+2)^2}(1)$$

$When\ t = 3$

$$\frac{dM}{dt} = \frac{4500}{(3(3)+2)^2}$$

$$= 37.2\ tonnes/year(1)$$

$$M = 900 - \frac{1500}{2 + 5\ln(t+1)}$$

$$M = 900 - 1500(2 + 5\ln(t+1))^{-1}$$

$$\frac{dM}{dt} = 1500(2 + 5\ln(t+1))^{-2}(1) \times \frac{5}{t+1}(1)$$

$$= \frac{7500}{(t+1)(2+5\ln(t+1))^2}(1)$$

$When\ t = 3$

$$\frac{dM}{dt} = \frac{7500}{(3+1)(2+5\ln(3+1))^2}$$

$= 23.5\ tonnes/year$(1) [8]

2. (i) $u(x,t) = e^{-k^2t}\sin x$

$\frac{\partial u}{\partial t} = -k^2e^{-k^2t}\sin x$ (1) [1]

(ii) $\frac{\partial u}{\partial x} = e^{-k^2t}\cos x$ (1)

$\frac{\partial^2 u}{\partial x^2} = -e^{-k^2t}\sin x$ (1) [2]

(b) $\frac{\partial u}{\partial t} = k^2\frac{\partial^2 u}{\partial x^2}$

$-k^2e^{-k^2t}\sin x = k^2 \times -e^{-k^2t}\sin x$ (1)

Yes the experimental equation satisfies the theoretical equation (1) [2]

3. (a) $\frac{d(xtan^{-1}x)}{dx} = x \times \frac{1}{1+x^2}(1) + \tan^{-1}x\ (1)$

$\frac{d(xtan^{-1}x)}{dx} = \frac{x}{1+x^2} + \tan^{-1}x$ (1) [3]

(b) $\int_0^1 \tan^{-1}x = \int_0^1 \frac{d(xtan^{-1}x)}{dx} - \int_0^1 \frac{x}{1+x^2}$ (1)

$$= [xtan^{-1}x]_0^1 (1) - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1 (1)$$

$$= [\tan^{-1}1 - 0] - \left[\frac{1}{2}\ln 2 - \frac{1}{2}\ln 1\right] (1)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2\ (1)$$

$$= \frac{\pi}{4} - \ln 2^{\frac{1}{2}}$$

$= \frac{\pi}{4} - \ln\sqrt{2}\quad as\ required(1)$ [6]

4. (a)

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	[2]
y	0	0.73508(1)	1.17157	1.02280(1)	0	

(b) $h = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}(1)$

$$\int_0^{\frac{\pi}{2}} \frac{2\sin 2x}{1+\cos x}\,dx$$

$= \frac{\frac{\pi}{8}}{2}[0 + 0 + 2(0.73508 + 1.17157 + 1.02280)]$(1)

$= 1.1504$ (1) [3]

(c) $u = 1 + \cos x$

$\frac{du}{dx} = -\sin x$ (1)

$$dx = \frac{du}{-\sin x}$$

$\cos x = u - 1$

$$\int \frac{2\sin 2x}{1+\cos x}\,dx = \int \frac{2\sin 2x}{u}\,\frac{du}{-\sin x} \text{(1)}$$

$$= \int \frac{4\sin x\cos x}{u}\,\frac{du}{-\sin x} \text{(1)}$$

$$= -\int \frac{4\cos x}{u}\,du$$

$$= -\int \frac{4(u-1)}{u}\,du \text{(1)}$$

$$= -4\int 1 - \frac{1}{u}\,du$$

$$= -4(u - \ln u) \text{(1)}$$

$$= -4u + 4\ln u$$

$$= -4(1+\cos x) + 4\ln(1+\cos x)$$

$$= 4\ln(1+\cos x) - 4\cos x - 4$$

$$+ c \text{ as required} \text{(1)} \quad [6]$$

(d) $\int_0^{\frac{\pi}{2}} \frac{2\sin 2x}{1+\cos x}\,dx$

$$= [4\ln(1+\cos x) - 4\cos x - 4]_0^{\frac{\pi}{2}}$$

$$= \left[4\ln\left(1+\cos\frac{\pi}{2}\right) - 4\cos\frac{\pi}{2} - 4\right] - [4\ln(1+\cos 0) - 4\cos 0 - 4] \text{(1)}$$

$$= 0 - 0 - 4 - 4\ln 2 + 4 + 4$$

$$= 4 - 4\ln 2 \text{ (1)} \quad [2]$$

(e) Difference $= 4 - 4\ln 2 - 1.1504$

$= 0.077$(1) [1]

5. (a) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$(1)

$(\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^3 + 4\cos\theta\,(i\sin\theta)^4 + (i\sin\theta)^4$(1)

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$ (1)

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ as required(1) [4]

(b)(i) $|z_1 + z_2| = |2 + 3i + 3 + 2i|$

$= |5 + 5i|$ (1)

$= \sqrt{5^2 + 5^2}$ (1)

$= \sqrt{50}$ (1)

$= 5\sqrt{2}$ (1) [4]

(ii) $w = \frac{z_1 z_3}{z_2}$

$$= \frac{(2+3i)(a+bi)}{3+2i} \text{ (1)}$$

$$= \frac{2a + 2bi + 3ai + 3bi^2}{3+2i} \text{ (1)}$$

$$= \frac{2a + 2bi + 3ai - 3b}{3+2i} \times \frac{3-2i}{3-2i} \text{(1)}$$

$$= \frac{6a + 6bi + 9ai - 9b - 4ai + 4b + 6a + 6bi}{9+4} \text{(1)}$$

$$= \frac{12a - 5b + 5ai + 12bi}{13} \text{ (1)}$$

$$= \frac{12a - 5b}{13} + \frac{5a + 12b}{13}i \text{ (1)} \quad [6]$$

(iii) $\frac{12a-5b}{13} + \frac{5a+12b}{13}i = \frac{17}{13} - \frac{7}{13}i$

$\frac{12a-5b}{13} = \frac{17}{13}$ (1) $\qquad \frac{5a+12b}{13} = -\frac{7}{13}$(1)

$12a - 5b = 17 \qquad 5a + 12b = -7$

$60a - 25b = 85$ (1)

$60a + 144b = -84$ (1)

$-169b = 169$

$b = -1$ (1)

$12a + 5 = 17$

$a = 1$ (1) [6]

(iv) $\arg w = \tan^{-1}\left(\frac{-\frac{7}{13}}{\frac{17}{13}}\right)$

$= \tan^{-1}\left(-\frac{7}{17}\right)$ (1)

$= -0.391\ rads$ (1) [2]