HARRISON COLLEGE INTERNAL EXAMINATION 2016
CARIBBEAN ADVANCED PROFICIENCY EXAMINATION
SCHOOL BASED ASSESSMENT
PURE MATHEMATICS
UNIT 2 – TEST 1
1 hour 20 minutes

This examination paper consists of 2 pages.
This paper consists of 3 questions.
The maximum marks for this examination is 60.

INSTRUCTIONS TO CANDIDATES

1. Write your name clearly on each sheet of paper used.
2. Answer **ALL** questions.
3. Do **NOT** do questions beside one another.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to **three** (3) significant figures.

EXAMINATION MATERIALS ALLOWED

1. Mathematical formulae sheet
2. Scientific Non-programmable calculator (non-graphical)

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1. (a) (i) Show that $\frac{dy}{dx} = -2$ at the point $A(1,1)$ on the curve

$$x + y - tan^{-1}(y) = 2 - \frac{1}{4}\pi,$$ [3]

(ii) Find the value of $\frac{d^2y}{dx^2}$ at A. [4]

(b) The curve C is defined parametrically by

$$x = t - \ln t\ , \quad y = 2(t-1) \quad t \in \mathbb{R} \quad t \geq 1$$

Find $\frac{d^2y}{dx^2}$ in terms of *t*. [5]

(c) Given that $z = e^{x^2 - y}$, show that

$$\frac{1}{z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$$ [5]

Total: 17 marks

PLEASE TURN OVER

2. (a) Find (i) $\int x^2 \sin\ x\ dx$ [5]

(ii) $\int \frac{x}{\sqrt{1-9x^4}}\ dx$ [4]

(b) Let $f(x) = \frac{3x^2-6x+2}{x(x-1)^2}$

i. Express $f(x)$ in partial fractions. [4]

ii. Hence find $\int_2^4 f(x)\ dx$ giving your answer in the form $\ln a - \frac{b}{c}$, where $a, b, c \in \mathbb{R}$. [5]

(c) Use the trapezium rule with 4 strips to find an approximation to

$$\int_{-1}^{1} \sqrt{\ln(2+x)}\ dx$$

giving your answer to 2 decimal places. [4]

Total: 22 marks

3. (a) Given $|z| = 2\sqrt{2}$ find the complex number *z* that satisfies the equation

$\frac{16}{z} - \frac{8}{z^*} = 2 - 9i$ [5]

(b) (i) Solve the equation $z^3 = 4 - 4\sqrt{3}i$

giving your answers in the form $re^{i\theta}$ where $r > 0$ and exact θ, $-\pi < \theta \leq \pi$. [6]

(ii) Illustrate your values from b (i) on an Argand diagram. [3]

(c) The point *P* represents a complex number *z* on an Argand diagram, where

$$|z+3| = 2|z-6i|$$

Show that the locus of *P* is a circle, stating the coordinates of the centre and the radius of this circle. [7]

Total: 21 marks

END OF EXAMINATION

SOLUTIONS

1. (a) (i) Differentiating implicitly:

$1+\frac{dy}{dx}+\frac{1}{1+y^2}\frac{dy}{dx}=0$ (i) 1 mark

Substitute $x=1 \quad y=1$ into (i) 1 mark

$\frac{dy}{dx}=-2$ 1 mark Total 3 marks

(ii) Make $\frac{dy}{dx}$ the subject of (i):

$\frac{dy}{dx}=-\frac{1+y^2}{y^2}=-\frac{1}{y^2}-1$ 1 mark

$\frac{d^2y}{dx^2}=2y^{-3}\frac{dy}{dx}$ [differentiating implicitly] 1 mark

So at $A(1,1)$ [substituting for y and $\frac{dy}{dx}$] 1 mark

$\frac{d^2y}{dx^2}=2(1)^{-3}(-2)=-4$ 1 mark Total 4 marks

(b) $\frac{dx}{dt}=1-\frac{1}{t}=\frac{t-1}{t}$ $\frac{dy}{dt}=2$ 1 mark + 1 mark

$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{2t}{t-1}$ 1 mark

$\frac{d^2y}{dx^2}=\frac{d(\frac{dy}{dx})}{dt}\div\frac{dx}{dt}$ [dividing by $\frac{dx}{dt}$] 1 mark

$\frac{d^2y}{dx^2}=\frac{-2}{(t-1)^2}\div\frac{t-1}{t}=\frac{-2t}{(t-1)^3}$ 1 mark Total 5 marks

(c) $\frac{\partial z}{\partial x}=e^{x^2-y}(2x)$ $\frac{\partial z}{\partial y}=e^{x^2-y}(-1)$ 1 mark + 1 mark

$\frac{\partial^2 z}{\partial y\partial x}=-e^{x^2-y}(2x)$ 1 mark

$\frac{1}{z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}=\frac{1}{e^{x^2-y}}\cdot e^{x^2-y}(2x)\cdot e^{x^2-y}(-1)=-e^{x^2-y}(2x)=\frac{\partial^2 z}{\partial y\partial x}$

[substituting and simplifying] 1 mark + 1 mark Total 5 marks

2. (a) (i) $\int x^2 \sin x \; dx = -x^2 \int \sin x \; dx - \int \int \sin x \frac{d(x^2)}{dx} \; dx$ 1 mark

$= -x^2 \cos x + \int 2x \cos x \, dx$ 1 mark

$= -x^2 \cos x + 2 \left[x \int \cos x - \int \int \cos x \frac{d(x)}{dx} \; dx \right]$ 1 mark

$= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx$ 1 mark

$= -x^2 \cos x + 2x \sin x + 2 \cos x + constant$ 1 mark Total 5 marks

(a) (ii) $\int \frac{x}{\sqrt{1-9x^4}} \; dx$, let $u = 3x^2$ 1 mark

$\frac{du}{dx} = 6x$

Substituting into given integral : $\frac{1}{6} \int \frac{1}{\sqrt{1-u^2}} \; dx$ 1 mark

$= \frac{1}{6} sin^{-1} u + constant$ 1 mark

$= \frac{1}{6} \; sin^{-1}(3x^2) + constant$ 1 mark Total 4 marks

(b) (i) $f(x) = \frac{3x^2-6x+2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$ 1 mark

$= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$

Let $x = 0 \Longrightarrow \; A = 2$ 1 mark

Let $x = 1 \Longrightarrow C = 3 - 6 + 2 = -1$ 1 mark

Consider x^2 terms: $A + B = 3 \; \Longrightarrow B = 1$ 1 mark

$f(x) = \frac{3x^2-6x+2}{x(x-1)^2} = \frac{2}{x} + \frac{1}{(x-1)} - \frac{1}{(x-1)^2}$ Total 4 marks

END OF EXAMINATION

(b) (ii) $\int_2^4 f(x)\ dx = \int_2^4 \frac{2}{x} + \frac{1}{(x-1)} - \frac{1}{(x-1)^2}\ dx$

$= \ 2\ln x + \ln|x-1| + (x-1)^{-1} \ \ |_2^4$ 1 mark + 1 mark + 1 mark

$= ln16 + ln3 + \frac{1}{3} - ln4 - ln1 - 1$ 1 mark

$= \ln 12 - \frac{2}{3}$ 1 mark Total 5 marks

(c) x $\sqrt{\ln(2+x)}$ [correct use of trapezium rule] 1 mark

x		$\sqrt{\ln(2+x)}$	
-1	0		[five ordinates used] 1 mark
-0.5		0.6377	
0		0.8326	[correct 'y" values] 1 mark
0.5		0.9572	
1	1.048		
$\sum =$	1.048	2.4275	

$\int_{-1}^{1} \sqrt{\ln(2+x)}\ dx = \frac{1}{2}\cdot\frac{1}{2}\left(1.048 + 2(\ 2.4275)\right) = 1.476\ \approx 1.48$ 1 mark Total 4 marks

3. (a) Let $z = a + ib$ $|z| = |a+ib| = \sqrt{a^2+b^2} = 2\sqrt{2}$

$a^2 + b^2 = 8$ 1 mark

$\frac{16}{z} - \frac{8}{z^*} = 2 - 9i$

$\frac{16}{(a+ib)} - \frac{8}{(a-ib)}$ 1 mark

$\frac{8a-24bi}{a^2+b^2}$ 1 mark

$\frac{8a-24bi}{8} = 2 - 9i$ 1 mark

$a = 2$ $b = 3$ 1 mark Total 5 marks

(b) $z^3 = 4 - 4\sqrt{3}i$ $\quad |4 - 4\sqrt{3}i| = \sqrt{16 + 16 \times 3} = \sqrt{64} = 8$ 1 mark

$\arg(4 - 4\sqrt{3}i) = -\frac{\pi}{3}\ rads$ 1 mark

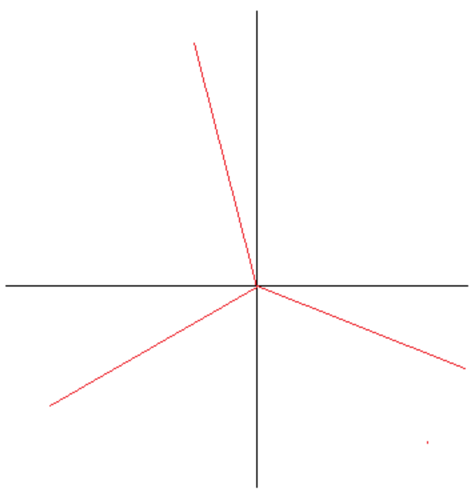
$z = (8e^{i(2n\pi - \frac{\pi}{3})})^{\frac{1}{3}} = 2e^{i(\frac{\pi}{9}(6n-1))}$ [using DeMorgan's rule] 1 mark

$n = 0 \quad z_1 = 2e^{-\frac{\pi}{9}i}$ 1 mark

$n = 1 \quad z_2 = 2e^{\frac{5\pi}{9}i}$ 1 mark

$n = -1 \quad z_3 = 2e^{-\frac{7\pi}{9}i}$ 1 mark Total 6 marks

(b) (ii)



Total 3 marks

END OF EXAMINATION

3. (c) $|z+3| = 2|z-6i|$

$z = x - iy$

$|x+iy+3| = \sqrt{(x+3)^2+y^2}$ 1 mark

$2|x+iy-6i| = 2\sqrt{x^2+(y-6)^2}$

$(x+3)^2+y^2 = 4(x^2+(y-6)^2)$ 1 mark

$x^2+6x+9+y^2 = 4x^2+4y^2-48y+144$

$3x^2-6x+3y^2-48y = -135$ 1 mark

$x^2-2x+y^2-16y = -45$ 1 mark

$x^2-2x+\ (-1)^2+y^2-16y+(-8)^2 = -45+1+64$

$(x-1)^2+(y-8)^2 = 20$ 1 mark

Equation of a circle with centre (1, 8) 1 mark

And radius = $\sqrt{20}$ units. 1 mark Total 7 marks