FORM TP 2016282

TEST CODE **02234020**

MAY/JUNE 2016

CARIBBEAN EXAMINATIONS COUNCIL

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 2 – Paper 02

ANALYSIS, MATRICES AND COMPLEX NUMBERS

2 hours 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Each section consists of TWO questions.
3. Answer ALL questions from the THREE sections.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra lined page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

SECTION A

Module 1

Answer BOTH questions.

1. (a) A quadratic equation is given by $ax^2 + bx + c = 0$, where $a, b, c \in \mathbf{R}$. The complex roots of the equation are $\alpha = 1 - 3i$ and β.

(i) Calculate $(\alpha + \beta)$ and $(\alpha\beta)$.

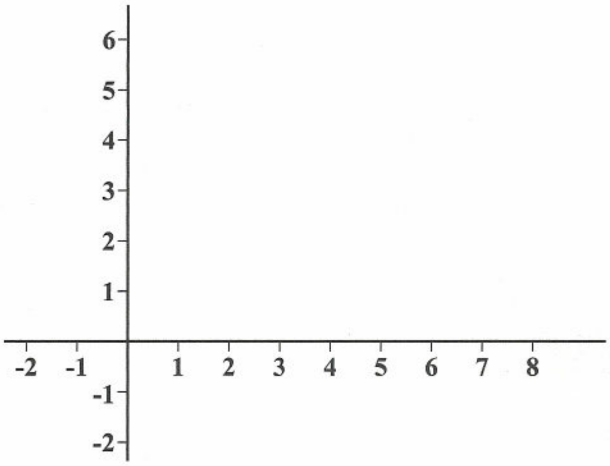
[3 marks]

(ii) Hence, show that an equation with roots $\frac{1}{\alpha - 2}$ and $\frac{1}{\beta - 2}$ is given by $10x^2 + 2x + 1 = 0$.

[6 marks]

(b) Two complex numbers are given as $u = 4 + 2\text{i}$ and $v = 1 + 2\sqrt{2}\text{i}$.

(i) Complete the Argand diagram below to illustrate u.



[1 mark]

(ii) On the same Argand plane, sketch the circle with equation $|z - u| = 3$.

[2 marks]

(iii) Calculate the modulus and principal argument of $z = \left(\frac{u}{v}\right)^5$.

[6 marks]

GO ON TO THE NEXT PAGE

(c) A function f is defined by the parametric equations

$$x = 4\cos t \text{ and } y = 3\sin 2t \text{ for } 0 \leq t \leq \pi.$$

Determine the x-coordinates of the two stationary values of f.

[7 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

2. (a) A function w is defined as $w\,(x, y) = \ln\left|\dfrac{2x + y}{x - 10}\right|$.

Determine $\dfrac{\partial w}{\partial x}$.

[4 marks]

(b) Determine $\int e^{2x} \sin e^{x} \, dx$.

[6 marks]

(c) Let $f(x) = \dfrac{x^2 + 2x + 3}{(x-1)(x^2+1)}$ for $2 \leq x \leq 5$.

(i) Use the trapezium rule with three equal intervals to estimate the area bounded by f and the lines $y = 0$, $x = 2$ and $x = 5$.

[5 marks]

(ii) Using partial fractions, show that $f(x) = \frac{3}{x-1} - \frac{2x}{x^2+1}$.

[6 marks]

GO ON TO THE NEXT PAGE

(iii) Hence, determine the value of $\int_{2}^{5} f(x)\,dx$.

[4 marks]

Total 25 marks

SECTION B

Module 2

Answer BOTH questions.

3. (a) A sequence is defined by the recurrence relation $u_{n+1} = u_{n-1} + x(u_n)'$, where $u_1 = 1$, $u_2 = x$ and $(u_n)'$ is the derivative of u_n.

For example, $u_3 = u_{2+1} = u_1 + x(u_2)' = 1 + x$.

Given that $u_8 = 13x + 1$ and that $u_{10} = 34x + 1$, find $(u_9)'$.

[4 marks]

(b) The n^{th} partial sum of a series, S_n, is given by $S_n = \sum_{r=1}^{n} r(r-1)$.

(i) Show that $S_n = \dfrac{n(n^2-1)}{3}$.

[7 marks]

(ii) Hence, or otherwise, evaluate $\sum_{10}^{20} r(r-1)$.

[5 marks]

(c) (i) Given that ${}^{n}P_{r} = \dfrac{n!}{(n-r)!}$ show that $\dfrac{{}^{2r}P_{r}\ {}^{n}P_{r}}{(2r)!}$ is equal to the binomial coefficient ${}^{n}C_{r}$.

[4 marks]

GO ON TO THE NEXT PAGE

(ii) Determine the coefficient of the term in x^3 in the binomial expansion of $(3x + 2)^5$.

[5 marks]

Total 25 marks

4. (a) The function f is defined as $f(x) = \sqrt[6]{4x^2 + 4x + 1}$ for $-1 < x < 1$.

(i) Show that $f(x) = (1 + 2x)^{\frac{1}{3}}$.

[3 marks]

The series expansion of $(1 + x)^k$ is given as

$$1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \ldots$$

where $k \in \mathbf{R}$ and $-1 < x < 1$.

(ii) Determine the series expansion of f up to and including the term in x^4.

[5 marks]

(iii) Hence, approximate f(0.4) correct to 2 decimal places.

[3 marks]

(b) The function $h(x) = x^3 + x - 1$ is defined on the interval [0, 1].

(i) Show that $h(x) = 0$ has a root on the interval [0, 1].

[3 marks]

(ii) Use the iteration $x_{n+1} = \dfrac{1}{x_n^2 + 1}$ with initial estimate $x_1 = 0.7$ to estimate the root of h correct to 2 decimal places.

(c) Use two iterations of the Newton–Raphson method with initial estimate $x_1 = 1$ to approximate the root of the equation $g(x) = e^{4x-3} - 4$ in the interval [1, 2]. Give your answer correct to 3 decimal places.

[5 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

SECTION C

Module 3

Answer BOTH questions.

5. (a) A bus has 13 seats for passengers. Eight passengers boarded the bus before it left the terminal.

(i) Determine the number of possible seating arrangements of the passengers who boarded the bus at the terminal.

[2 marks]

(ii) At the first stop, no passengers will get off the bus but there are eight other persons waiting to board the same bus. Among those waiting are three friends who must sit together.

Determine the number of possible groups of five of the waiting passengers that can join the bus.

[4 marks]

(b) Gavin and his best friend Alexander are two of the five specialist batsmen on his school's cricket team.

Given that the specialist batsmen must bat before the non-specialist batsmen and that all five specialist batsmen may bat in any order, what is the probability that Gavin and Alexander are the opening pair for a given match?

[5 marks]

(c) A matrix **A** is given as

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -1 & 6 & 0 \end{bmatrix}$$

(i) Find the |**A**|, determinant of **A**.

[4 marks]

(ii) Hence, or otherwise, find $\mathbf{A}^{-1}$, the inverse of **A**.

[10 marks]

Total 25 marks

6. (a) Two fair coins and one fair die are tossed at the same time.

(i) Calculate the number of outcomes in the sample space.

[3 marks]

(ii) Find the probability of obtaining exactly one head.

[2 marks]

(iii) Calculate the probability of obtaining at least one head on the coins and an even number on the die on a particular attempt.

[4 marks]

GO ON TO THE NEXT PAGE

(b) Determine whether $y = C_1x + C_2x^2$ is a solution to the differential equation

$$\frac{x^2}{2}y'' - xy' + y = 0, \text{ where } C_1 \text{ and } C_2 \text{ are constants.}$$

[6 marks]

(c) (i) Show that the general solution to the differential equation

$$3\,(x^2 + x)\,\frac{\mathrm{d}y}{\mathrm{d}x} = 2y\,(1 + 2x) \text{ is}$$

$$y = C\,\sqrt[3]{(x^2 + x)^2}, \text{ where } C \in \mathbf{R}$$

[7 marks]

(ii) Hence, given that $y(1) = 1$, solve $3(x^2 + x)\frac{dy}{dx} = 2y(1 + 2x)$.

[3 marks]

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.