CAPE UNIT 2 2017

QUESTION 1

(a) $\frac{d}{dx}(\cos^{-1}(\sin^{-1}x))$

$$= -\frac{1}{\sqrt{1-(\sin^{-1}x)^2}} \times \frac{1}{\sqrt{1-x^2}}$$

(b) (i) $w(x,y) = \ln\left|\frac{2x+y}{x-1}\right|$

$$= \ln|2x+y| - \ln|x-1|$$

$$\frac{\partial w}{\partial x} = \frac{2}{2x+y} - \frac{1}{x-1}$$

$$-\frac{1}{9} = \frac{2}{2(4)+y_0} - \frac{1}{4-1}$$

$$\frac{2}{9} = \frac{2}{8+y_0}$$

$$9 = 8 + y_0$$

$$y_0 = 1$$

(ii) $w(x,y) = \ln|2x+y| - \ln|x-1|$

$$\frac{\partial w}{\partial y} = \frac{1}{2x+y}$$

$$\frac{\partial^2 w}{\partial y \partial x} = -\frac{2}{(2x+y)^2}$$

$$\frac{\partial^2 w}{\partial y^2} = -\frac{1}{(2x+y)^2}$$

$$\frac{\partial^2 w}{\partial y \partial x} - 2\frac{\partial^2 w}{\partial y^2} = -\frac{2}{(2x+y)^2} - 2\left(-\frac{1}{(2x+y)^2}\right)$$

$$= -\frac{2}{(2x+y)^2} + \frac{2}{(2x+y)^2}$$

$$= 0$$

(c) (i) $u^2 = -15 + 8i$

$$(x+iy)^2 = -15 + 8i$$

$$x^2 - y^2 + 2xyi = -15 + 8i$$

$$x^2 - y^2 = -15$$

$$xy = 4 \quad \rightarrow \quad x = \frac{4}{y}$$

$$\left(\frac{4}{y}\right)^2 - y^2 = -15$$

$$\frac{16}{y^2} - y^2 + 15 = 0$$

$$y^4 - 15y^2 - 16 = 0$$

$$(y^2-16)(y^2+1) = 0$$

$$y^2 = 16$$

$$y = \pm 4$$

$$x = \pm 1$$

$$u = \pm(1+4i)$$

(ii) $z^2 - (3+2i)z + (5+i) = 0$

$$z = \frac{(3+2i) \pm \sqrt{(3+2i)^2 - 4(1)(5+i)}}{2(1)}$$

$$z = \frac{(3+2i) \pm \sqrt{5+12i-20-4i}}{2}$$

$$z = \frac{(3+2i) \pm \sqrt{-15+8i}}{2}$$

$$z = \frac{(3+2i) \pm (1+4i)}{2}$$

$$z = \frac{4+6i}{2} = 2+3i$$

$$z = \frac{2-2i}{2} = 1-i$$

QUESTION 2

(a) (i) $I_n = \int x^n e^{ax}\ dx$

$u = x^n$ $\quad\quad$ $du = nx^{n-1}$

$dv = e^{ax}$ $\quad\quad$ $v = \frac{1}{a}e^{ax}$

$$\int x^n e^{ax}\ dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\ dx$$

$aI_n = x^n e^{ax} - nI_{n-1}$

(ii) $\int x^3 e^{3x}\ dx$

$$I_3 = \frac{1}{3}[x^3e^{3x} - 3I_2]$$

$$I_3 = \frac{1}{3}\left[x^3e^{3x} - 3\left[\frac{1}{3}(x^2e^{3x} - 2I_1)\right]\right]$$

$$I_3 = \frac{1}{3}[x^3e^{3x} - x^2e^{3x} + 2I_1]$$

$$I_3 = \frac{1}{3}\left[x^3e^{3x} - x^2e^{3x} + 2\left(\frac{1}{3}(xe^{3x} - I_0)\right]\right.$$

$$I_3 = \frac{1}{3}\left[x^3e^{3x} - x^2e^{3x} + \frac{2}{3}xe^{3x} - \frac{2}{3}I_0\right]$$

$$I_3 = \frac{1}{3}\left[x^3e^{3x} - x^2e^{3x} + \frac{2}{3}xe^{3x} - \frac{2}{3}\int e^{3x}\ dx\right]$$

$$I_3 = \frac{1}{3}x^3e^{3x} - \frac{1}{3}x^2e^{3x} + \frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x} + c$$

(b) $\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\ dx$

$$= \left[\frac{(\sin^{-1}x)^2}{2}\right]_0^1$$

$$= \left(\frac{(\sin^{-1}(1))^2}{2}\right) - \left(\frac{(\sin^{-1}(0))^2}{2}\right)$$

$$= \frac{\pi^2}{8}$$

(c) (i) $\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)}$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$

When $x = 0$

$4 = 4A$

$1 = A$

Coefficients of x

$C = -1$

Coefficients of x^2

$2 = A + B$

$2 = 1 + B$

$B = 1$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}$$

$$= \frac{1}{x} + \frac{x}{x^2 + 4} - \frac{1}{x^2 + 4}$$

(ii) $\int \frac{2x^2-x+4}{x^3+4x}\, dx$

$$= \int \frac{1}{x} dx + \frac{1}{2}\int \frac{2x}{x^2+4}\, dx - \frac{1}{2}\int \frac{2}{x^2 + (2)^2}\, dx$$

$$= \ln x + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

QUESTION 3

(a) (i) $f(x) = \ln(5 + x)$ $\quad f(2) = \ln 7$

$f'(x) = \frac{1}{5+x}$ $\quad f(2) = \frac{1}{7}$

$f''(x) = -\frac{1}{(x+5)^2}$ $\quad f''(2) = -\frac{1}{49}$

$f'''(x) = \frac{2}{(x+5)^3}$ $\quad f'''(2) = \frac{2}{343}$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$\ln(x+5) = \ln 7 + \frac{1}{7}(x-2) - \frac{1}{98}(x-2)^2 + \frac{1}{1029}(x-2)^3$$

(ii) $f(7) - \ln 7$

$$\ln(12) - \ln(7) = \frac{1}{7}(7-2) - \frac{1}{98}(7-2)^2 + \frac{1}{1029}(7-2)^3 = 0.5807$$

(b) (i) $P_n: \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$P_1: 1^3 = \frac{1}{4}(1)^2(1+1)^2$

$1 = 1$

Therefore P_1 is true

Assume P_n is true for $n = k$

$$P_k : \sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$$

$$P_{k+1} : \sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2$$

$P_{k+1} = P_k + (k+1)\text{term}$

$$= \frac{k^2}{4}(k+1)^2 + (k+1)^3$$

$$= \frac{k^2}{4}(k+1)^2 + \frac{4(k+1)^3}{4}$$

$$= \frac{(k+1)^2}{4}[k^2 + 4(k+1)]$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

Therefore P_{k+1} is true $\forall$ P_k is true.

Hence by mathematical induction $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ for $n \in \mathbb{N}$.

(ii) $\sum_{i=1}^{2n+1} i^3$

$$= \frac{1}{4}(2n+1)^2(2n+2)^2$$
$$= \frac{1}{4}(2n+1)^2[2(n+1)]^2$$
$$= \frac{1}{4}(2n+1)^2(4(n+1)^2)$$
$$= (2n+1)^2(n+1)^2$$

(iii) $\sum_{i=1}^{n+1}(2i-1)^3 = 1^3 + 3^3 + 5^3 + \cdots + (2n+1)^3$

$$\sum_{i=1}^{n+1}(2i-1)^3 = \sum_{i=1}^{2n+1} i^3 - \sum_{i=1}^{n}(2i)^3$$

$$\sum_{i=1}^{n+1}(2i-1)^3 = \sum_{i=1}^{2n+1} i^3 - 8\sum_{i=1}^{n} i^3$$

$$= (2n+1)^2(n+1)^2 - 8\left(\frac{1}{4}n^2(n+1)^2\right)$$

$$= (2n+1)^2(n+1)^2 - 2n^2(n+1)^2$$

$= (n+1)^2[(2n+1)^2 - 2n^2]$

$= (n+1)^2(4n^2 + 4n + 1 - 2n^2)$

$= (n+1)^2(2n^2 + 4n + 1)$

QUESTION 4

(a) Arrange 8 boys = 8!

Arrange 2 girls in 7 available spaces (i.e. between the boys) = ${}^7P_2$

Number of arrangements with girls not together or at ends = 8! x ${}^7P_2$ = 1 693 440

(b) (i) $\left(1 + \frac{1}{8}x\right)^8 = \binom{8}{0}\left(\frac{x}{8}\right)^0 + \binom{8}{1}\left(\frac{x}{8}\right)^1 + \binom{8}{2}\left(\frac{x}{8}\right)^2 + \binom{8}{3}\left(\frac{x}{8}\right)^3 + \binom{8}{4}\left(\frac{x}{8}\right)^4$

$= 1 + 8\left(\frac{x}{8}\right) + 28\left(\frac{x^2}{64}\right) + 56\left(\frac{x^3}{512}\right) + 70\left(\frac{x^4}{4096}\right)$

$= 1 + x + \frac{7}{16}x^2 + \frac{7}{64}x^3 + \frac{35}{2048}x^4$

(ii) $(1.0125)^8$

$\left(1 + \frac{1}{8}(0.1)\right)^8 = 1 + 0.1 + \frac{7}{16}(0.1)^2 + \frac{7}{64}(0.1)^3 + \frac{35}{2048}(0.1)^4 = 1.104486$

(c) (i) $f(x) = \sqrt{x} - \cos x$

$f(0) = \sqrt{0} - \cos(0) = -1$

$f(1) = \sqrt{1} - \cos(0) = 1$

$f(x)$ is continuous on the interval $[0, 1]$

(ii) Mid – point of the interval $[0, 1]$ is 0.5

$f(0.5) = \sqrt{0.5} - \cos(0.5) = -0.5$

Root in interval $[0.5, 1]$

Mid – point of interval $[0.5, 1]$ is 0.75

Root = 0.75

(d) (i) $-2x^3 - 3x + 9 = 0$

$2x^3 = 9 - 3x$

$x^3 = \frac{9 - 3x}{2}$

$x = \sqrt[3]{\frac{9 - 3x}{2}}$

$x_{n+1} = \sqrt[3]{\frac{9 - 3x_n}{2}}$

(ii) $x_1 = 1$

$x_2 = \sqrt[3]{\frac{9 - 3(1)}{2}} = 1.4422$

$x_3 = 1.32698$

QUESTION 5

(a) (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.7 = 0.6 + 0.2 - P(A \cap B)$

$P(A \cap B) = 0.1$

$P(A \text{ only}) = 0.6 - 0.1 = 0.5$

(ii) two events are independent if

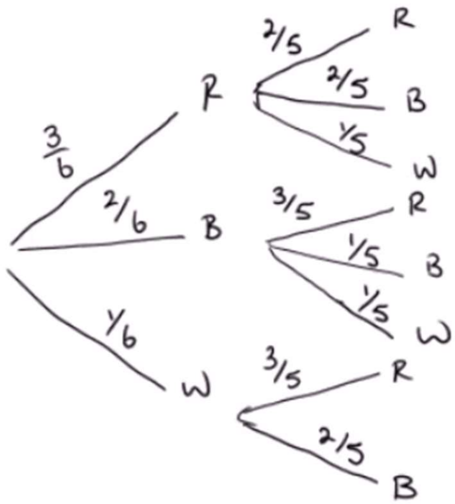
$P(A \cap B) = P(A)P(B)$

$P(A \cap B) = 0.1$

$P(A)P(B) = (0.6)(0.2) = 0.12$

Therefore the two events are not independent

(b) (i)



(ii) $P(1^{st}\ Red) \times P(2^{nd}\ White) + P(1^{st}\ Blue) \times P(2^{nd} White)$

$= \frac{3}{6} \times \frac{1}{5} + \frac{2}{6} \times \frac{1}{5} = \frac{1}{6}$

(c) (i) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 3 & 9 \end{pmatrix}\begin{pmatrix} 30 & -12 & 2 \\ 5 & -8 & 3 \\ -5 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix} = 20I$

$AB = 20I$

$A\left(\frac{1}{20}B\right) = I$

(ii) $A^{-1} = \frac{1}{20}\begin{pmatrix} 30 & -12 & 2 \\ 5 & -8 & 3 \\ -5 & 4 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 3 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 25 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 30 & -12 & 2 \\ 5 & -8 & 3 \\ -5 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \\ 25 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 20 \\ 40 \\ 40 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

QUESTION 6

(a) (i) $(1 + x^2)\frac{dy}{dx} + 2xy = \sqrt[3]{x}$

We have an exact differential equation

$$\int \left((1 + x^2)\frac{dy}{dx} + 2xy\right) dx = \int x^{\frac{1}{3}}\, dx$$

$$y(1 + x^2) = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c$$

$$y(1 + x^2) = \frac{3x^{\frac{4}{3}}}{4} + c$$

(ii) when $y = 2, x = 0$

$$2(1 + 0^2) = \frac{3(0)^{\frac{4}{3}}}{4} + c$$

$$2 = c$$

$$y(1 + x^2) = \frac{3x^{\frac{4}{3}}}{4} + 2$$

$$y = \frac{3x^{\frac{4}{3}}}{4(1 + x^2)} + \frac{2}{1 + x^2}$$

$$y(1) = \frac{3(1)^{\frac{4}{3}}}{4(1 + 1^2)} + \frac{2}{1 + 1^2} = \frac{11}{8}$$

(b) (i) $u = y'$

$u' = y''$

$y'' + 4y' = 2\cos 3x - 4\sin 3x$

$u' + 4u = 2\cos 3x - 4\sin 3x$

(ii) Auxiliary equation:

$m + 4 = 0$

$m = -4$

Complementary function: $u = Ae^{-4x}$

Particular Integral

Let $u = m\sin 3x + n\cos 3x$

$u' = 3m\cos 3x - 3n\sin 3x$

$3m\cos 3x - 3n\sin 3x + 4(m\sin 3x + n\cos 3x) = 2\cos 3x - 4\sin 3x$

$(3m + 4n)\cos 3x + (4m - 3n)\sin 3x = 2\cos 3x - 4\sin 3x$

Equating coefficients of $\cos 3x$

$3m + 4n = 2 \qquad (1)$

Equating coefficients of $\sin 3x$

$4m - 3n = -4 \qquad (2)$

Solving (1) and (2) simultaneously:

$m = -\frac{2}{5}$

$n = \frac{4}{5}$

Particular Integral: $u = -\frac{2}{5}\sin 3x + \frac{4}{5}\cos 3x$

General solution is $u = Ae^{-4x} - \frac{2}{5}\sin 3x + \frac{4}{5}\cos 3x$

Since $u = y'$

$y = -\frac{1}{4}Ae^{-4x} + \frac{2}{15}\cos 3x + \frac{4}{15}\sin 3x + c$