# HARRISON COLLEGE INTERNAL EXAMINATION 2022
## CARIBBEAN ADVANCED PROFICIENCY EXAMINATION
## SCHOOL BASED ASSESSMENT
## PURE MATHEMATICS
## UNIT 2 – TEST 1
**1 hour 20 minutes**

This examination paper consists of 14 pages.
This paper consists of 3 questions.
The maximum marks for this examination is 60.

**INSTRUCTIONS TO CANDIDATES**

1. Write your name clearly on each sheet of paper used.
2. Answer **ALL** questions.
3. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to **three** (3) significant figures.

**EXAMINATION MATERIALS ALLOWED**

1. Mathematical formulae sheet
2. Scientific Non-programmable calculator (non-graphical)

---

1. (a) (i) Differentiate $f(x) = cos^{-1}3x$ where $-\frac{1}{3} < x < \frac{1}{3}$. [2]

$$f'(x) = \frac{-3}{\sqrt{1-(3x)^2}} = \frac{-3}{\sqrt{1-9x^2}}$$

(ii) Differentiate $y = \frac{\ln 3x}{sin^{-1}x}$ [3]

$$\frac{dy}{dx} = \frac{(\sin^{-1}x)\,\frac{1}{3x}\cdot 3 - \ln 3x\,\frac{1}{\sqrt{1-x^2}}}{(\sin^{-1}x)^2}$$

$$= \frac{\frac{1}{x}\sin^{-1}x - \frac{\ln 3x}{\sqrt{1-x^2}}}{(\sin^{-1}x)^2}$$

PLEASE TURN OVER

(b) The curve C is defined parametrically by

$$x = t + \ln(t+1)\ , \quad y = 3te^{2t}$$

Find the equation of the tangent to the curve at the origin. [5]

$$\frac{dx}{dt} = 1 + \frac{1}{t+1} = \frac{t+2}{t+1}$$

$$\frac{dy}{dt} = 3te^{2t}\cdot 2 + e^{2t}(3)$$

$$= 3e^{2t}(2t+1)$$

$$\frac{dy}{dx} = 3e^{2t}(2t+1)\cdot\frac{t+1}{t+2}$$

at origin $x=0$ $y=0$

$0 = t + \ln(t+1) \Rightarrow t = 0$

$\frac{dy}{dx}$ (at origin i.e. $t=0$)

$$= 3\cdot 1\cdot 1 \times \frac{1}{2} = \frac{3}{2}$$

equation of tangent

$$y = \frac{3}{2}x$$

PLEASE TURN OVER

(c) Consider the curve defined by $y^2 = \sin(xy), y \neq 0$, show that $\frac{dy}{dx} = \frac{y\cos(xy)}{2y - x\cos(xy)}$. [5]

differentiating implicitly

$$2y\frac{dy}{dx} = \cos xy\left(x\frac{dy}{dx} + y\right)$$

$$2y\frac{dy}{dx} = x\cos xy\frac{dy}{dx} + y\cos(xy)$$

$$2y\frac{dy}{dx} - x\cos xy\frac{dy}{dx} = y\cos(xy)$$

$$\frac{dy}{dx}(2y - x\cos xy) = y\cos(xy)$$

$$\frac{dy}{dx} = \frac{y\cos xy}{2y - x\cos xy}$$

(d) Let $f(x,y) = x^2y - 2x + y^3$, find $\frac{\partial^2 f}{\partial x \partial y}$. [2]

$$\frac{\partial F}{\partial y} = x^2 + 3y^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x$$

**Total: 17 marks**

PLEASE TURN OVER

2. (a) Find $\int \frac{1}{\sqrt{16-6x-x^2}}\,dx$ [3]

$$-x^2 - 6x + 16 = 25 - (x+3)^2 \checkmark$$

$$\int \frac{1}{\sqrt{16-6x-x^2}}\,dx = \int \frac{1}{\sqrt{25-(x+3)^2}}\,dx = \int \frac{1}{\sqrt{5^2-(x+3)^2}}\,dx \checkmark$$

$$= \sin^{-1}\left(\frac{x+3}{5}\right) + c \checkmark$$

PLEASE TURN OVER

(b) Let $f(x) = \frac{6x^2+8x+9}{(2-x)(3+2x)^2}$

i. Express $f(x)$ in partial fractions. [5]

$$\frac{6x^2+8x+9}{(2-x)(3+2x)^2} = \frac{A}{2-x} + \frac{B}{3+2x} + \frac{C}{(3+2x)^2}$$

$$= \frac{A(3+2x)^2 + B(2-x)(3+2x) + C(2-x)}{(2-x)(3+2x)^2}$$

let $x=2$ : $49A = 49 \Rightarrow A = 1$

Consider coefficient of $x^2$

: $4A - 2B = 6 \Rightarrow B = -1$

Consider constants

$9A + 6B + 2C = 9 \Rightarrow C = 3$

$$f(x) = \frac{1}{2-x} - \frac{1}{3+2x} + \frac{3}{(3+2x)^2}$$

**PLEASE TURN OVER**

ii. Hence find $\int_{-1}^{0} f(x)\ dx$ giving your answer in the form $a + \frac{1}{2}\ln(\frac{b}{c})$, where $a, b, c \in \mathbb{R}$. [5]

$$\int_{-1}^{0} f(x)\,dx = \int_{-1}^{0} \frac{1}{2-x} - \frac{1}{3+2x} + \frac{3}{(3+2x)^2}\,dx$$

$$= -\ln|2-x| - \frac{\ln|3+2x|}{2} - \frac{3}{2(3+2x)}\Bigg|_{-1}^{0}$$

$$= \left[-\ln 2 - \frac{1}{2}\ln 3 - \frac{1}{2}\right] - \left[-\ln 3 - \frac{\ln 1}{2} - \frac{3}{2}\right]$$

$$= 1 - \ln 2 + \frac{1}{2}\ln 3$$

$$= 1 - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 3$$

$$= 1 + \frac{1}{2}(\ln 3 - \ln 4)$$

$$= 1 + \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

PLEASE TURN OVER

(c) It is given that for $n \geq 0$

$$I_n = \int_0^e x(\ln x)^n \, dx$$

(i) Prove that for $n \geq 1$, $2I_n = e^2 - nI_{n-1}$. [4]

$$I_n = (\ln x)^n \int x \, dx - \int \int x \, n(\ln x)^{n-1} \frac{1}{x} \, dx \Big|_0^e$$

$$= (\ln x)^n \frac{x^2}{2} - \frac{1}{2} n \int_0^e x (\ln x)^{n-1} \, dx \Big|_0^e$$

$$= \frac{e^2}{2} - \frac{1}{2} n I_{n-1}$$

$$2I_n = e^2 - nI_{n-1}$$



7

(ii) Find the exact value of $I_3$ . [4]

$$I_3 = \frac{e^2}{2} - \frac{3}{2} I_2$$

$$I_2 = \frac{e^2}{2} - I_1$$

$$I_1 = \frac{e^2}{2} - \frac{1}{2} I_0$$

$$I_0 = \int_0^e x \, dx = \frac{e^2}{2}$$

$$I_1 = \frac{e^2}{2} - \frac{1}{2}\frac{e^2}{2} = \frac{e^2}{4}$$

$$I_2 = \frac{e^2}{2} - \frac{e^2}{4} = \frac{e^2}{4}$$

$$I_3 = \frac{e^2}{2} - \frac{3}{2}\frac{e^2}{4} = \frac{e^2}{8}$$



8

(d) Use the trapezium rule with 4 strips to find an approximation to

$$\int_{-1}^{1} \sqrt{\ln(2+x)}\, dx$$

giving your answer to 2 decimal places. [4]

$h = \frac{1-(-1)}{4} = \frac{1}{2}$

| $x$ | $\sqrt{\ln(2+x)}$ | |
|---|---|---|
| $-1$ | $0$ | |
| $-0.5$ | | $0.6368$ |
| $0$ | | $0.8326$ |
| $0.5$ | | $0.9572$ |
| $1$ | $1.048$ | |
| $\Sigma$ | $1.048$ | $2.4266$ |

$\int_{-1}^{1} \sqrt{\ln(2+x)}\, dx = \frac{1}{2} \cdot \frac{1}{2}\left(1.048 + 2(2.4266)\right)$

$= 1.4753$

$= 1.48$

**Total: 25 marks**

3. (a) Showing all necessary working, express the complex number $\frac{2+3i}{1-2i}$ in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$ .

Give the value of $r$ and $\theta$ correct to 3 significant figures. [5]

$$z = \frac{2+3i}{1-2i} \cdot \frac{1+2i}{1+2i} = -\frac{4}{5} + \frac{7}{5}i$$

$$r = \sqrt{\left(-\frac{4}{5}\right)^2 + \left(\frac{7}{5}\right)^2} = \sqrt{\frac{65}{25}} = \sqrt{\frac{13}{5}} = 1.61$$

$$\arg. z = \theta = \pi - \tan^{-1}\left(\frac{7/5}{4/5}\right) = 2.09^c$$

$$z = 1.61\, e^{2.09i}$$

(b) The complex number $2 - i$ is denoted by $u$.

It is given that $u$ is root of the equation $x^3 + ax^2 - 3x + b = 0$, where the constants $a$ and $b$ are real. Find the values of $a$ and $b$. [4]

substituting $x = 2 - i$

$(2-i)^3 + a(2-i)^2 - 3(2-i) + b = 0$

$(2 - 11i) + a(3 - 4i) - 6 + 3i + b = 0$ ✓

$\left.\begin{array}{l} 3a + b - 4 = 0 \\ -11 - 4a + 3 = 0 \end{array}\right\}$ ✓ $\Rightarrow$ $a = -2$ ✓

$b = 10$ ✓

---

Alternatively

$\left.\begin{array}{l} \alpha = 2 - i \\ \beta = 2 + i \\ \gamma = K \quad K \in \mathbb{R} \end{array}\right\}$ ✓

$\left.\begin{array}{l} \alpha + \beta + \gamma = -a = 4 + K = -a \\ \alpha\beta\gamma = -b = 5 \times K = b \end{array}\right\}$ ✓

$\left.\begin{array}{l} \alpha\beta + \beta\gamma + \gamma\alpha = -3 \\ 5 + K(4) = -3 \\ K = -2 \end{array}\right\}$ ✓

$\left.\begin{array}{l} 4 + (-2) = -a \Rightarrow a = -2 \\ 5 + (-2) = b \Rightarrow b = 10 \end{array}\right\}$ ✓

PLEASE TURN OVER

(c) The complex number $u$ is given by $u = -1 + (4\sqrt{3})i$.

Find the two square roots of $u$.

Give your answers in the form $a + ib$, where $a$ and b are exact. [5]

let $z = (x + iy) = \sqrt{-1 + 4\sqrt{3}\,i}$

$z^2 = (x+iy)^2 = x^2 - y^2 + 2xy\,i = -1 + 4\sqrt{3}\,i$

$x^2 - y^2 = -1$

$2xy = 4\sqrt{3}$ ✓

$xy = 2\sqrt{3}$

$y = \frac{2\sqrt{3}}{x} \Rightarrow y^2 = \frac{12}{x^2}$ ✓

$x^2 - \frac{12}{x^2} = -1$

$x^4 - 12 = -x^2$

$x^4 + x^2 - 12 = 0$ ✓

$(x^2 + 4)(x^2 - 3)$

$x^2 = 3$

$x = \pm\sqrt{3}$ ✓

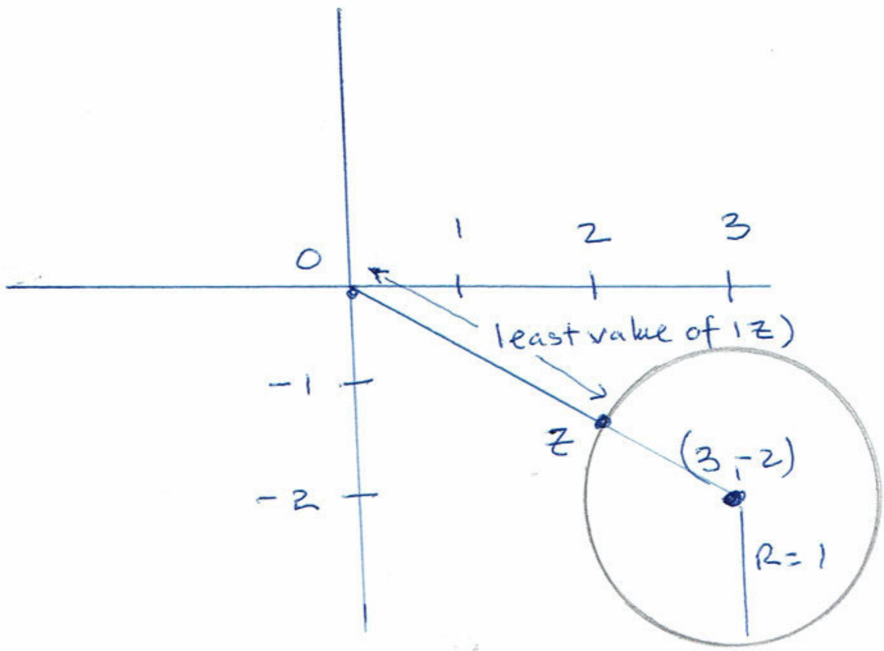
when $x = +\sqrt{3} \Rightarrow y = 2$

$x = -\sqrt{3} \Rightarrow y = -2$ ✓

So roots of $u$ are

$(\sqrt{3} + 2i)$ and $-\sqrt{3} - 2i$

(d) On an Argand diagram sketch the locus of points representing complex numbers $z$ satisfying the equation $|z - 3 + 2i| = 1$. Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [4]




locus — circle
centre $(3, -2)$
radius $(1)$

least value of $|z| = \sqrt{(3)^2 + (-2)^2} - 1$
$= \sqrt{13} - 1$

**Total: 18 marks**



**END OF EXAMINATION**