CARIBBEAN ADVANCED PROFICIENCY EXAMINATION

SCHOOL BASED ASSESSMENT

PURE MATHEMATICS

UNIT 2 – TEST 1

Time: 1 Hour & 20 minutes

This examination paper consists of 3 printed pages.

The paper consists of 7 questions.

The maximum mark for this examination is 60.

INSTRUCTIONS TO CANDIDATES

1. Write your name clearly on each sheet of paper used.
2. Answer **ALL** questions.
3. Number your questions carefully and do **NOT** write your solutions to different questions beside one another.
4. Unless otherwise stated in the question, any numerical answer that is not exact, **MUST** be written correct to three (3) significant figures.

EXAMINATION MATERIALS ALLOWED

1. Mathematical formulae
2. Electronic calculator (non – programmable, non – graphical)

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1. Determine $\frac{\partial f}{\partial x}$ in each of the following cases.

(i) $f(x, y) = 3x^2y + e^{x+y} - \sin xy$ [3]

(ii) $f(x, z) = \ln xz^2 \tan^{-1}(xz)$ [4]

Total 7 Marks

2. The complex number, z, is such that $z = -5 + 12i$. Determine

(i) $|z|$, [1]

(ii) $\arg z$, [2]

(iii) $\frac{z}{4-3i}$ [4]

Total 7 Marks

3. Let $z = \cos\theta + i\sin\theta$, then

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

(i) Express $16\sin^5\theta$ in the form $\sin 5\theta + p\sin 3\theta + q\sin\theta$, where p and q are integers to be determined.

[6]

(ii) Hence, determine the exact value of

$$\int_0^{\frac{\pi}{3}} 16\sin^5\theta \; d\theta$$

[4]

Total 10 Marks

4. Given that $\tan y = x$, for $x > 0$.

(i) Use implicit differentiation to show that

$$\frac{d^2y}{dx^2} = -2x\left(\frac{dy}{dx}\right)^2$$

[4]

(ii) Hence find the value of $\frac{d^2y}{dx^2}$ at the point $\left(1, \frac{\pi}{4}\right)$. [2]

Total 6 Marks

5. A curve has parametric equations

$$x = 2\cot t\,, \quad y = 2\sin^2 t\,, \quad 0 < t < \frac{\pi}{2}.$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t. [4]

(b) Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$. [4]

Total 8 Marks

6. Let

$$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \; dx$$

(i) Prove that, for $n \geq 2$,

$$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

[8]

(ii) Calculate the exact value of I_1 and deduce the exact value of I_3. [4]

Total 12 Marks

7. Given that

$$I = \int_1^5 x \ln x \; dx$$

(i) determine the exact value of I using integration by parts. [5]

(ii) use the trapezium rule with 4 trapezia to obtain an estimate of I. [5]

Total 10 Marks

SOLUTIONS

1. (i)	$f(x,y) = 3x^2y + e^{x+y} - \sin xy$ $\frac{\partial f}{\partial x} = 6xy + e^{x+y} - y\cos xy$	1 mark each for the correct implicit differentiation of each term
(ii)	$f(x,z) = \ln xz^2 \tan^{-1}(xz)$ $\frac{\partial f}{\partial x} = \frac{1}{x}(\tan^{-1}(xz)) + \ln xz^2 \left(\frac{z}{1+(xz)^2}\right)$ $\frac{\partial f}{\partial x} = \frac{\tan^{-1}(xz)}{x} + \frac{z \ln xz^2}{1+(xz)^2}$	1 – attempting to use the product rule 1 – implicit derivative of $\ln xz^2$ 1 – implicit derivative of $\tan^{-1}(xz)$ 1 - C.A.O
	Total	**7 Marks**
2. (i)	$\lvert z\rvert = \sqrt{(-5)^2 + 12^2} = 13$	1 – C.A.O
(ii)	$\arg z = \pi - \tan^{-1}\left(\frac{12}{5}\right) = 1.967^c$	1 – correct formula for the argument 1 – C.A.O (degrees or radians)
(iii)	$\frac{-5+12i}{4-3i}$ $= \frac{(-5+12i)(4+3i)}{(4-3i)(4+3i)}$ $= \frac{-20+48i-15i+36i^2}{16+9}$ $= \frac{-56+33i}{25}$ $= -\frac{56}{25} + \frac{33}{25}i$	1 – use of the conjugate of the denominator 1 – evaluation of the numerator 1 – evaluation of the denominator 1 – writing answer in the form $a + bi$
	Total	**7 Marks**
3. (i)	$\left(z - \frac{1}{z}\right)^5 = \left(z + \left(-\frac{1}{z}\right)\right)^5$ $= z^5 + 5z^4\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2 + 10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$ $= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$	1 – correct expansion of $\left(z - \frac{1}{z}\right)^5$ 1 – correct coefficients for the expansion 1 – correct grouping of z^5, z^3 and z 1 – expansion of $(2i\sin\theta)^5$

	$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$ $(2i \sin\theta)^5$ $32i^5 \sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$ $32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$ $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$	1 – substitution of $2i\sin n\theta$ 1 – simplification of $16\sin^5\theta$
(ii)	$\int_0^{\frac{\pi}{3}} 16\sin^5\theta \; d\theta$ $= \int_0^{\frac{\pi}{3}} \sin 5\theta \; d\theta - 5\int_0^{\frac{\pi}{3}} \sin 3\theta \; d\theta + 10\int_0^{\frac{\pi}{3}} \sin\theta \; d\theta$ $= \left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_0^{\frac{\pi}{3}}$ $= \left[-\frac{\cos 5\left(\frac{\pi}{3}\right)}{5} + \frac{5\cos 3\left(\frac{\pi}{3}\right)}{3} - 10\cos\left(\frac{\pi}{3}\right)\right] - \left[-\frac{\cos 0}{5} + \frac{5\cos 0}{3} - 10\cos\theta\right]$ $= \left[-\frac{1}{10} - \frac{5}{3} - 5\right] - \left[-\frac{1}{5} + \frac{5}{3} - 10\right]$ $= \frac{53}{30}$	1 – substituting $\sin 5\theta - 5\sin 3\theta + 10\sin\theta$ for $16\sin^5\theta$ 1 – integrating $\sin 5\theta - 5\sin 3\theta + 10\sin\theta$ 1 – use of limits 1 – C.A.O
	Total	**10 Marks**
4. (i)	$\tan y = x$ $\sec^2 y \frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{1}{\sec^2 y} = (\sec y)^{-2}$ $\frac{d^2y}{dx^2} = -2(\sec y)^{-3}(\sec y \tan y)\frac{dy}{dx}$ $= -\frac{2}{\sec^2 y}\tan y\left(\frac{1}{\sec^2 y}\right)$ $\frac{d^2y}{dx^2} = -2x\left(\frac{1}{\sec^2 y}\right)\left(\frac{1}{\sec^2 y}\right)$	1 – differentiating $\tan y$ with respect to x 1 – determining $\frac{dy}{dx}$ 1 – determining $\frac{d^2y}{dx^2}$ 1 – substituting $\frac{dy}{dx} = \left(\frac{1}{\sec^2 y}\right)$ into $\frac{d^2y}{dx^2}$

	$\frac{d^2y}{dx^2} = -2x\left(\frac{dy}{dx}\right)^2$	
(ii)	At $\left(1, \frac{\pi}{4}\right)$ $\frac{d^2y}{dx^2} = -2(1)\left(\left(\cos\frac{\pi}{4}\right)^2\right)^2$ $\frac{d^2y}{dx^2} = -\frac{1}{2}$	1 – subbing $x = 1$ and $y = \frac{\pi}{4}$ into $\frac{d^2y}{dx^2}$ 1 - his correct value for $\frac{d^2y}{dx^2}$
	Total	**6 Marks**
5. (i)	$x = 2\cot t$ $\frac{dx}{dt} = -2\csc^2 t = -\frac{2}{\sin^2 t}$ $y = 2\sin^2 t = 2(\sin t)^2$ $\frac{dy}{dt} = 4\sin t\cos t$ $\frac{dy}{dx} = 4\sin t\cos t \times \frac{\sin^2 t}{-2} = -2\sin^3 t\cos t$	1 – determining $\frac{dx}{dt}$ 1 – determining $\frac{dy}{dt}$ 1 – use of $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (S.O.I) 1 – his expression for $\frac{dy}{dx}$
(ii)	When $t = \frac{\pi}{4}$ $x = 2\cot\left(\frac{\pi}{4}\right) = 2$ $y = 2\left(\sin\left(\frac{\pi}{4}\right)\right)^2 = 1$ $\frac{dy}{dx} = -2\left(\sin\left(\frac{\pi}{4}\right)\right)^3\cos\left(\frac{\pi}{4}\right) = -\frac{1}{2}$ $y = mx + c$ $1 = -\frac{1}{2}(2) + c$ $c = 2$ $y = -\frac{1}{2}x + 2$	1 – determining the value of $\frac{dy}{dx}$ 1 – determining the value of x 1 – determining the value of y 1 – determining the equation
	Total	**8 Marks**

6. (i)	$$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \ dx$$ $u = x^n \quad \rightarrow \quad du = nx^{n-1}$ $dv = \sin x \quad \rightarrow \quad v = -\cos x$ $$I = [-x^n \cos x]_0^{\frac{\pi}{2}} + n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x \ dx$$ $$I_n = n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x \ dx$$ $u = x^{n-1} \quad \rightarrow \quad du = (n-1)x^{n-2}$ $dv = \cos x \quad \rightarrow \quad v = \sin x$ $$I_n = n\left[[x^{n-1} \sin x]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \sin x \, dx\right]$$ $$I_n = n\left[\left(\frac{\pi}{2}\right)^{n-1} - (n-1)I_{n-2}\right]$$ $$I_n + (n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$	1 – use of u and dv 1 – use of integration by parts (S.O.I) 1 - $[-x^n \cos x]_0^{\frac{\pi}{2}} = 0$ 1 – use of u and dv 1 – use of integration by parts 1 – $[x^{n-1} \sin x]_0^{\frac{\pi}{2}} = \left(\frac{\pi}{2}\right)^{n-1}$ 1 – $\int_0^{\frac{\pi}{2}} x^{n-2} \sin x \, dx = I_{n-2}$ 1 – Rearranging equation to required form
(ii)	$$I_1 = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$ $$I_1 = \left(\frac{\pi}{2}\right)^0 - 1(1-1)I_{-2}$$ $$I_1 = 1$$ $$I_3 = 3\left(\frac{\pi}{2}\right)^{3-1} - 3(3-1)I_1$$ $$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 6I_1$$ $$I_3 = \frac{3\pi^2}{4} - 6$$ $$I_3 = \frac{3\pi^2 - 24}{4}$$	1 – evaluating I_1 1 – Subbing 3 into I_3 1 – Subbing I_1 into I_3 1 - $I_3 = \frac{3\pi^2-24}{4}$ or equivalent
	Total	**12 Marks**

7. (i)	$I = \int_1^5 x \ln x \; dx$ $u = \ln x \quad \to \quad du = \frac{1}{x}$ $dv = x \quad \to \quad v = \frac{x^2}{2}$ $I = \left[\frac{x^2}{2} \ln x\right]_1^5 - \int_1^5 \frac{1}{x}\left(\frac{x^2}{2}\right) dx$ $I = \left[\frac{5^2}{2} \ln 5\right] - \left[\frac{x^2}{2}\right]_1^5$ $I = \frac{25 \ln 5}{2} - \left(\frac{5^2}{2} - \frac{1^2}{2}\right)$ $I = \frac{25 \ln 5}{2} - 6$	1 – use of $u = \ln x$ and $dv = x$ 1 – Integrating $x \ln x$ by parts 1 – integrating $\frac{x}{2}$ 1 – use of limits 1 – C.A.O
(ii)	$y_0 = 0$ $y_1 = 2 \ln 2$ $y_2 = 3 \ln 3$ $y_3 = 4 \ln 4$ $y_4 = 5 \ln 5$ $I = \frac{5-1}{2(4)}[(0 + 5 \ln 5) + 2(2 \ln 2 + 3 \ln 3 + 4 \ln 4)]$ $= 14.251$	1 – determining $n = 1$ 1 – evaluating $y_0, y_1, y_2,$ 1 – evaluating y_3, y_4 1 – substituting values into Trapezium Rule formula 1 – C.A.O
	Total	**10 Marks**