HARRISON COLLEGE INTERNAL EXAMINATION 2021

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION

SCHOOL BASED ASSESSMENT

PURE MATHEMATICS

UNIT 1 – TEST 3

Time: 1 hour and 20 minutes

NAME OF STUDENT: ____________ SOLUTIONS

SCHOOL CODE: 030014__________

DATE: ______________________________________

This examination paper consists of 9 printed pages and 1 blank page.

The paper consists of 3 questions.

The maximum mark for this examination is 60.

INSTRUCTIONS TO CANDIDATES

1. Write your name clearly in the space above.
2. Answer **ALL** questions in the **SPACES PROVIDED**.
3. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided. You must also write your name and candidate number clearly on any additional paper used.
4. Number your questions **carefully and identically to those on the question paper**.
5. Unless otherwise stated in the question, any numerical answer that is not exact, **MUST** be written correct to three (3) significant figures.

EXAMINATION MATERIALS ALLOWED

1. Mathematical formulae
2. Electronic calculator (non-programmable, non-graphical)

1. Evaluate

(a)

(i) $\lim_{x\to 3} \frac{2x^2-5x-3}{x-3}$ [3]

$\lim_{x\to 3} \frac{(2x+1)(x-3)}{x-3}$ [1]

$\lim_{x\to 3} 2x + 1$ [1]

$= 2(3) + 1$

$= 7$ [1]

(ii) $\lim_{x\to 0} \frac{\sin x}{\sin 2x}$ [5]

$= \lim_{x\to 0} \left[\frac{\sin x}{x} \times \frac{x}{sin2x}\right]$ [1]

$= \lim_{x\to 0} \left[\frac{\sin x}{x} \times \frac{2x}{2sinx}\right]$ [1]

$= \lim_{x\to 0} \left[\frac{1}{2} \times \frac{\sin x}{x} \times \frac{2x}{sinx}\right]$

$= \frac{1}{2} \times 1 \times 1$ [1]

$= \frac{1}{2}$ [1]

(b) The function f on $\mathbb{R}$ is defined by,

$$f(x) = \begin{cases} 3x - 7, & x > 4 \\ 1 + 2x, & x \leq 4 \end{cases}$$

(i) Find

$$\lim_{x \to 4^+} f(x)$$ [2]

$$\lim_{x \to 4^+} 3x - 7$$ [1]

$$= 3(4) - 7$$

$$= 5$$ [1]

(ii) Find

$$\lim_{x \to 4^-} f(x)$$ [2]

$$\lim_{x \to 4^-} 1 + 2x$$ [1]

$$= 1 + 2(4)$$

$$= 9$$ [1]

(iii) Deduce that $f(x)$ is discontinuous a $x = 4$. [2]

$$\mathit{Since} \lim_{x \to 4^+} f(x) \neq \lim_{x \to 4^-} f(x)$$ [1]

$\mathit{then} \lim_{x \to 4} f(x)$ *does not exist. Hence* $f(x)$ *is discontinuous at* $x = 4$. [1]

2. From first principles, find the values of the derivative of the function:

$$f(x) = x^2 + 2x$$ [5]

$$f(x+h) = (x+h)^2 + 2(x+h)$$

$$\lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{x \to 0} \frac{(x+h)^2 + 2(x+h) - x^2 - 2x}{h}$$ [1] + [1]

$$\lim_{x \to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h}$$ [1]

$$\lim_{x \to 0} \frac{2xh + h^2 + 2h}{h}$$

$$\lim_{x \to 0} \frac{h(2x + h + 2)}{h}$$

$$= 2x + 0 + 2$$ [1]

$$= 2x + 2$$ [1]

(b) Show that the differential of $y = x^2 sinx$ with respect to x is $x(2\sin x + xcosx)$ [3]

[1] [1]

$$\frac{dy}{dx} = x^2 cosx + sinx \cdot 2x$$

$$= x^2 cosx + 2xsinx$$

$$= x(xcosx + 2sinx)$$ [1]

(c) P is the point on the curve $y = 2x^3 + kx - 5$ where $x = 1$ and the gradient is -2. Find:

(i) the value of k. [3]

$$\frac{dy}{dx} = -2$$

$\frac{dy}{dx} = 6x^2 + k$ [1]

$\therefore\ 6x^2 + k = -2$ [1]

$when\ x = 1$

$6 + k = -2$

$k = -8$ [1]

(ii) the value of $\frac{d^2y}{dx^2}$ at P. [2]

$$\frac{dy}{dx} = 6x^2 - 8$$

$\frac{d^2y}{dx^2} = 12x$ [1]

$when\ x = 1$

$\frac{d^2y}{dx^2} = 12$ [1]

(iii) the equation of the normal to the curve at P. [4]

the gradient at P = -2

$\therefore$ *gradient of normal is* $\frac{1}{2}$ [1]

$when\ x = 1; y = 2(1)^3 - 8(1) - 5$
$= -11$ [1]

$y = mx + c$

$-11 = \frac{1}{2}(1) + c$

$-11 - \frac{1}{2} = c$

$\rightarrow c = -11.5$ [1]

$\therefore y = \frac{1}{2}x - 11.5$ [1]

(c) Amelia, an engineer, was tasked to construct a roller coaster at a popular amusement park in Florida. The path of the roller coaster is represented by the equation:

$$y = x^3 - 3x + 2$$

(i) Find $\frac{dy}{dx}$ in terms of x. [1]

$$\frac{dy}{dx} = 3x^2 - 3$$ [1]

(ii) Using your solution to part (i), find the coordinates of the stationary points of the roller coaster. [4]

At stationary points, $\frac{dy}{dx} = 0$

$\therefore 3x^2 - 3 = 0$ [1]

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$ [1]

When $x = 1; y = 1^3 - 3(1) + 2$
$= 0$

When $x = -1; y = (-1)^3 - 3(-1) + 2$
$= 4$

Hence the stationary points are $(1,0)$ *and* $(-1,4)$. [1] + [1]

(iii) Determine when the roller coaster:

a. soars to its maximum point

b. plunges to its lowest point [6]

$$\frac{d^2y}{dx^2} = 6x$$

Maximum point occurs when $\frac{d^2y}{dx^2} < 0$. [1]

So, $\left.\frac{d^2y}{dx^2}\right|_{-1} = 6(-1)$

$= -6$ [1]

$-6 < 0$

Therefore a maximum point occurs at $(-1,4)$ [1]

Minimum point occurs when $\frac{d^2y}{dx^2} > 0$. [1]

So, $\left.\frac{d^2y}{dx^2}\right|_{1} = 6(1)$

$= 6$ [1]

$6 > 0$

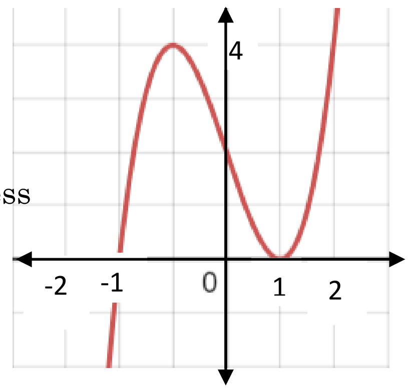
Therefore a maximum point occurs at $(1,0)$ [1]

(iv) Sketch the curve $y = x^3 - 3x + 2$, for $-2 \le x \le 2$, showing clearly, the maximum and minimum points of the roller coaster. [3]

[1] – max point

[1] – min point

[1] – tidiness/smoothness



3. By using the substitution $u = x^2 + 1$, evaluate $\int_0^{\sqrt{2}} \frac{4x}{x^2+1}\, dx$. [6]

$\frac{du}{dx} = 2x$ [1]

$du = 2xdx$

When $x = 0\,,\ u = 1$ [1]

$x = \sqrt{2}, u = 3$ [1]

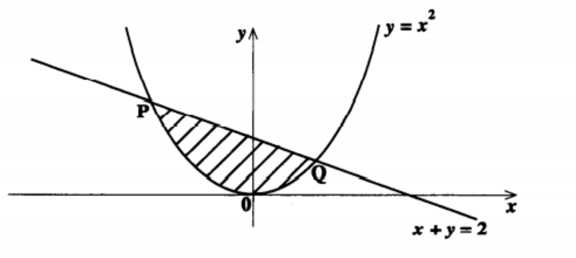
$\int_0^{\sqrt{2}} \frac{4x}{x^2+4} = \int_1^3 \frac{2}{\sqrt{u}} du$ [1]

$= \left[4\sqrt{u}\right]_1^3$ [1]

$= 4\sqrt{3} - 4\sqrt{1}$

$= 2.9$ [1]

(b) The diagram below shows the graphs of $y = x^2$ and $x + y = 2$.



(i) Find the coordinates of P and Q. [5]

$x^2 = 2 - x$ [1]
$x^2 + x - 2 = 0$
$(x-1)(x+2) = 0$ [1] + [1] – factorize and solve
$x = 1 \; or \; x = -2$

$When\; x = 1, \quad y = 1$
$When\; x = -2, y = 4$

$\therefore P(-2, 4) \; and \; Q\; (1,1)$ [1] + [1]

(ii) Find the area of the shaded region [4]

$$\text{Area} = \int_{-2}^{1}(x^2 + x - 2)\,dx$$

$$= \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x\right]_{-2}^{1} \qquad [1]$$

$$= -\frac{7}{6} - \frac{10}{3} \qquad [1] + [1]$$

$$= -\frac{9}{2} \qquad [1]$$