HARRISON COLLEGE INTERNAL EXAMINATION 2021

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION

SCHOOL BASED ASSESSMENT

PURE MATHEMATICS

UNIT 1 – TEST 2

Time: 1 hour and 20 minutes

NAME OF STUDENT: ____

SCHOOL CODE: 030014

DATE: ____________________________________

This examination paper consists of 9 printed pages.

The paper consists of 4 questions.

The maximum mark for this examination is 60.

INSTRUCTIONS TO CANDIDATES

1. Write your name clearly in the space above.
2. Answer **ALL** questions in the **SPACES PROVIDED**.
3. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided. You must also write your name and candidate number clearly on any additional paper used.
4. Number your questions **carefully and identically to those on the question paper**.
5. Unless otherwise stated in the question, any numerical answer that is not exact, **MUST** be written correct to three (3) significant figures.

EXAMINATION MATERIALS ALLOWED

1. Mathematical formulae
2. Electronic calculator (non-programmable, non-graphical).

1. a) Prove that

$$\sin\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)+\cos\left(\frac{\pi}{4}+\theta\right)\sin\left(\frac{\pi}{4}-\theta\right)=1$$

1 1 1

$$LHS=\sin\left(\frac{\pi}{4}+\theta+\frac{\pi}{4}-\theta\right)=\sin\left(\frac{\pi}{2}\right)=1$$

Total 3 marks

Alternatively, if the same result is obtained by expanding the formulae, then full marks are to be awarded.

[3 marks]

b) Show that

$$4+(\tan x-\cot x)^2\equiv cosec^2x+sec^2x$$

$$LHS=4+(\tan x-\cot x)^2$$

1 for expanding brackets

$$=4+tan^2x-2\tan x\cot x+cot^2x$$

$$=4+tan^2x+cot^2x-2\frac{\sin x}{\cos x}\times\frac{\cos x}{\sin x}$$

1 for expressing in terms of $\sin x$ *and* $\cos x$

$$=4-2+(sec^2x-1)+(cosec^2x-1)$$

$$=cosec^2x+sec^2x=RHS$$

1 for identity for tan^2x

1 for identity for cot^2x

Total 4 marks

[4 marks]

c) Find the general solution of the equation

$$6\,cos^2x + \sin x = 4$$

$$6\,cos^2x + \sin x = 4$$

$$6(1 - sin^2x) + \sin x = 4$$

$$6 - 6\,sin^2x + \sin x = 4$$

$$6\,sin^2x - \sin x - 2 = 0$$ 1

$$(2\sin x + 1)(3\sin x - 2) = 0$$

$$\sin x = -\frac{1}{2} \quad OR \quad \sin x = \frac{2}{3}$$ 1 1

$$sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \;\; or - 0.524 \qquad sin^{-1}\left(\frac{2}{3}\right) = 0.730$$ 1 1

General solution $x = n\pi + (-1)^n(-0.524)$ 1

$x = n\pi + (-1)^n(0.730)$ 1

Total 7 marks

[7 marks]

d) i) Express $2\cos x - 7\sin x$ in the form $r\cos(x+\alpha)$ where $r > 0$ *and* $0^0 \leq x \leq 90^0$.

$$2\cos x - 7\sin x = r\,cos(x+\alpha)$$

$$2\cos x - 7\sin x = r[\cos x\cos\alpha - \sin x\sin\alpha]$$

1 $2 = r\cos\alpha \qquad 7 = r\sin\alpha$ 1

$$4 = r^2cos^2\alpha \qquad 49 = r^2sin^2\alpha$$

Adding gives $53 = r^2cos^2\alpha + r^2sin^2\alpha$

$$53 = r^2(cos^2\alpha + sin^2\alpha)$$

$$r = \sqrt{53}$$ 1

$$\frac{r\sin\alpha}{r\cos\alpha} = \frac{7}{2} \qquad \tan\alpha = \frac{7}{2}, \alpha = tan^{-1}\left(\frac{7}{2}\right) = 74.1^0$$ 1

$$2\cos x - 7\sin x = \sqrt{53}\,cos(x + 74.1^0)$$ Total 4 marks

[4 marks]

ii) Hence, solve the equation $2\cos x - 7\sin x = -5, for\ 0^0 \leq x \leq 360^0$.

$$2\cos x - 7\sin x = -5$$

$$\sqrt{53}\, cos(x + 74.1^0) = -5$$

$$\cos(x + 74.1^0) = -\frac{5}{\sqrt{53}}$$ 1

cos is negative so $x + 74.1^0$ lies in the 2nd or 3rd quadrants

$$cos^{-1}\frac{5}{\sqrt{53}} = 46.6^0$$ 1

$$x + 74.1^0 = 180^0 - 46.6^0, 180^0 + 46.6^0$$

1

$$x + 74.1^0 = 133.4^0, 226.6^0$$ 1

$$x = 133.4^0 - 74.1^0, 226.6^0 - 74.1^0$$

$$= 59.3^0, 152.5^0$$

1 1

Total 6 marks

[6 marks]

Total 24 marks

2. A circle, C, has equation $x^2 + y^2 + 2x - 8y = 152$.

i) Find the radius and the coordinates of the centre of the circle.

$$x^2 + 2x + y^2 - 8y = 152$$

$$(x+1)^2 - 1 + (y-4)^2 - 16 = 152$$

$$(x+1)^2 + (y-4)^2 = 13^2$$ 1

$centre\ of\ circle\ (-1, 4)$ $radius = 13\ units$

1 for $x - coordinate$

1 for $y - coordinate$

1

Total 4 marks

[4 marks]

ii) Find the equation of the tangent to the circle at the point $P(4, 16)$.

$$\text{gradient of normal at } P(4,16) = \frac{16-4}{4-(-1)} = \frac{12}{5}$$ 1

$$\text{gradient of tangent} = -\frac{5}{12}$$ 1

$$\text{equation of tangent is } y = -\frac{5}{12}x + c$$

$$\text{sub } x = 4, y = 16 \text{ to find the value of } c$$

$$16 = -\frac{5}{12}(4) + c, \qquad c = \frac{53}{3}$$ 1

$$\text{Equation of tangent} \quad y = -\frac{5}{12}x + \frac{53}{3} \qquad OR \qquad 12y = -5x + 212$$

1

Total 4 marks

[4 marks]

iii) The point P is at one end of the diameter of the circle C. Find the coordinates of the point Q, at the opposite end of the diameter from P.

$$\text{Let } Q \text{ have coordinates } (a, b) \qquad \text{Since } P(4,16), \text{the mid} - \text{point can be expressed as}$$

$$(-1, 4) = \left(\frac{4+a}{2}, \frac{16+b}{2}\right)$$ 1

$$-1 = \frac{4+a}{2}, \quad a = -6 \qquad \text{and} \qquad 4 = \frac{16+b}{2}, \quad b = -8$$

1 1

Total 3 marks

[3 marks]

Total 11 marks

3. a) A curve, C, has parametric equations

$$x = sin^2\theta \; and \; y = \cos\theta\, , 0 \leq \theta < \frac{\pi}{2},$$

Find the Cartesian equation of C.

$$x = sin^2\theta \qquad y = \cos\theta$$

$$y^2 = cos^2\theta$$ 1

$$x + y^2 = sin^2\theta + cos^2\theta$$ 1

$$x + y^2 = 1$$

$$y^2 = 1 - x \quad OR \quad y = \sqrt{1 - x}$$

1

Total 3 marks

[3 marks]

b) Find the points of intersection of the curve

$$C_1 \; with \; parametric \; equations \; x = 4t^2 \; and \; y = 8t$$

and the line

$$L_1 \; with \; Cartesian \; equation \; 3y + 16 = 4x$$

$$x = 4t^2 \qquad y = 8t$$

$$\frac{y}{8} = t$$

$$x = 4\left(\frac{y}{8}\right)^2, \quad x = \frac{y^2}{16}$$ 1

$$3y + 16 = 4x, substituting \; for \; x \; we \; get \quad 3y + 16 = 4\left(\frac{y^2}{16}\right)$$ 1 for substitution

$$4(3y + 16) = y^2$$

$$y^2 - 12y - 64 = 0$$ 1

$$(y - 16)(y + 4) = 0$$

$$y = 16$$ 1

$$when \; y = 16, x = \frac{16^2}{16} = 16$$ 1

$$y = -4$$ 1

$$when \; y = -4, x = \frac{(-4)^2}{16} = 1$$ 1

Points of intersection $(16, 16) \quad and \quad (1, -4)$

Total 7 marks

Total 10 marks

4. a) Find the angle between the lines with equations

$$r = \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} \ and \ \frac{x+5}{2} = \frac{y-3}{1} = \frac{z+1}{-1}$$

Direction vector is $\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$ 1

$$Let \quad \mu = \frac{x+5}{2} \qquad \mu = \frac{y-3}{1} \qquad \mu = \frac{z+1}{-1}$$

$$x = -5 + 2\mu \qquad y = 3 + \mu \qquad z = -1 - \mu$$

$$r = \begin{pmatrix} -5 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \qquad direction\ vector\ is \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$ 1

$$\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$8 - 2 - 3 = \sqrt{16+4+9}\sqrt{4+1+1}\cos\theta$$

1 1

$$\frac{3}{\sqrt{29 \times 6}} = \cos\theta$$

$$\theta = cos^{-1}\left(\frac{3}{\sqrt{174}}\right) = 76.9^0$$

1

Total 5 marks

[5 marks]

b) Find the point of intersection of the lines with equations

$$r_1 = \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \quad and \quad r_2 = \begin{pmatrix} -6 \\ 17 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+3\mu \\ 7-2\mu \\ -1-2\mu \end{pmatrix} \quad \boxed{1} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6+\lambda \\ 17-2\lambda \\ -3+2\lambda \end{pmatrix}$$

$$3 + 3\mu = -6 + \lambda \quad multiply\ entire\ equation\ by\ 2\ \ to\ get\ 6 + 6\mu = -12 + 2\lambda$$

$$7 - 2\mu = 17 - 2\lambda$$

Solving the equations simultaneously by adding

$$13 + 4\mu = 5 \quad \boxed{1}$$

$$4\mu = -8, \quad \mu = -2 \quad \boxed{1}$$

Sub $\mu = -2$ into one of the equations, $3 + 3(-2) = -6 + \lambda$

$$\lambda = 3 \quad \boxed{1}$$

Use the equation for the $z - direction$ as a check

$$-1 - 2\mu = -3 + 2\lambda$$

$$-1 - 2(-2) = -3 + 2(3)$$

$$3 = 3\ so\ equations\ are\ consistent \quad \boxed{1}$$

To find the point of intersection sub $\mu = -2 \quad OR \quad \lambda = 3$

$$(3 + 3(-2), 7 - 2(-2), -1 - 2(-2)) = (-3, 11, 3)$$

$$\boxed{1}$$

$$OR\ (-6 + 3, 17 - 2(3), -3 + 2(3)\ ) = (-3, 11, 3)$$

Total 6 marks

[6 marks]

c) Find the Cartesian equation of the plane passing through the point $A(4, 2, -7)$ and parallel to the plane with equation $3x + 5y - z = 7$.

$$3x + 5y - z = 7$$

$$\boldsymbol{r}.\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = 7 \quad \textit{where the vector normal to the plane is} \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$ 1

$$\textit{Therefore a vector that is normal to a parallel plane is} \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

Vector equation of the plane is

$$\boldsymbol{r}.\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -7 \end{pmatrix}.\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$ 1

$$\boldsymbol{r}.\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = 12 + 10 + 7$$

$$\boldsymbol{r}.\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = 29$$ 1

Cartesian equation of plane is $3x + 5y - z = 29$ 1

Total 4 marks

[4 marks]

Total 15 marks

END OF TEST