HARRISON COLLEGE INTERNAL EXAMINATION, APRIL 2018

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION

SCHOOL BASED ASSESSMENT

PURE MATHEMATICS

UNIT 1 – TEST 3

Time: 1 Hour & 20 minutes

This examination paper consists of 3 printed pages.

The paper consists of 7 questions.

The maximum mark for this examination is 60.

<u>INSTRUCTIONS TO CANDIDATES</u>

1. Write your name clearly on each sheet of paper used.
2. Answer **ALL** questions.
3. Number your questions carefully and do **NOT** write your solutions to different questions beside one another.
4. Unless otherwise stated in the question, any numerical answer that is not exact, **MUST** be written correct to three (3) significant figures.

<u>EXAMINATION MATERIALS ALLOWED</u>

1. Mathematical formulae
2. Electronic calculator (non – programmable, non – graphical)

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1. (a) Determine

(i) $\lim_{x\to 2} \frac{x^3 - 4x}{x - 2}$ [3]

(ii) $\lim_{x\to 0} \frac{\sin 3x}{4x}$ [4]

(b) Find the values of x for which $\frac{x^2+1}{|3x+2|-8}$ is discontinuous. [4]

(c) A function $f(x)$ is defined as

$$f(x) = \begin{cases} x + 6 & x \leq 3 \\ x^2 & x > 3 \end{cases}$$

(i) Find $\lim_{x \to 3} f(x)$. [2]

(ii) Determine whether $f(x)$ is continuous at $x = 3$. Give a reason for your answer. [4]

(d) Differentiate $f(x) = \cos 2x$ using first principles. [6]

Total 23 Marks

2. Given that $y = 16x + \frac{1}{x}$, determine the equation of the tangent to the curve at the point where $x = 1$. [6]

TOTAL 6 Marks

3. The curve C has equation $y = \frac{x}{4+x^2}$.

(i) Show that $\frac{dy}{dx} = \frac{4-x^2}{(4+x^2)^2}$ [3]

(ii) Determine the coordinates of the stationary points on C. [4]

Total 7 Marks

4.

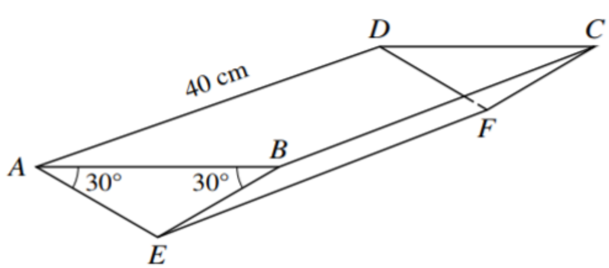


Fig. 1

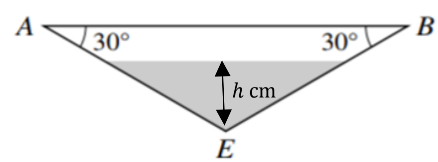


Fig. 2

Fig. 1 shows an open tank in the shape of a triangular prism. The vertical ends ABE and DCF are identical isosceles triangles, angle ABE = angle $BAE = 30°$. The length of AD is 40 cm. The tank is fixed in position with the open top $ABCD$ horizontal. Water is poured into the tank at a constant rate of 200 cm^3s^{-1}. The depth of water, t seconds after filling starts, is h cm (see Fig. 2).

(i) Show that, when the depth of water in the tank is h cm, the volume, V cm^3, of water in the tank is given by $V = (40\sqrt{3})h^2$. [3]

(ii) Find the rate at which h is increasing when $h = 5$. [3]

Total 6 Marks

5. The parametric equations of a curve are given by

$$x = \sin\theta\,,\, y = \cos 2\theta\,, \qquad 0 \leq \theta \leq 2\pi$$

Show that $\frac{dy}{dx} = -4\sin\theta$. [4]

Total 4 Marks

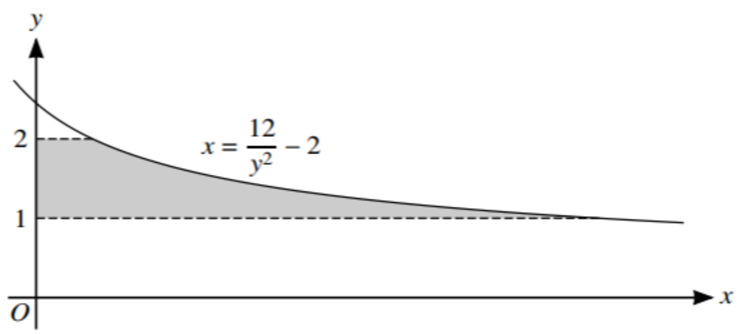
6. Use the substitution $u = \sin x + 1$ to show that

$$\int \sin x \cos x\,(1 + \sin x)^5\ dx = \frac{1}{42}(1 + \sin x)^6(6\sin x - 1) + c$$

[8]

Total 8 Marks

7.



The diagram shows part of the curve $x = \frac{12}{y^2} - 2$. The shaded region is bounded by the curve, the y – axis and the lines $y = 1$ and $y = 2$. Find the volume, in terms of π, when the shaded region is rotated through 360° about the y – axis. [6]

Total 6 Marks

END OF EXAMINATION

CAPE UNIT 1 TEST 3 MARK SCHEME

1. (i) $\lim_{x\to 2}\frac{x^3-4x}{x-2}$	
$=\lim_{x\to 2}\frac{x(x^2-4)}{x-2}$	1 - Completely factoring x^3-4x
$=\lim_{x\to 2}\frac{x(x+2)(x-2)}{x-2}$	1 – cancelling $x-2$
$=\lim_{x\to 2}x(x+2)$	
$=2(2+2)$	1 – C.A.O
$=8$	
(ii) $\lim_{x\to 0}\frac{\sin 3x}{4x}$	1 – Separation of terms
$=\lim_{x\to 0}\frac{\sin 3x}{3x}\times\frac{3x}{4x}$	1 – Use of appropriate property of limits (S.O.I)
$=\frac{3}{4}\lim_{x\to 0}\frac{\sin 3x}{3x}$	
$=\frac{3}{4}\times 1$	1 – Use of $\lim_{x\to 0}\frac{\sin x}{x}=1$
$=\frac{3}{4}$	1 – C.A.O
(b) $\lvert 3x+2\rvert-8=0$	1 – equating denominator to 0
$\lvert 3x+2\rvert=8$	
$3x+2=8$	1 – for $3x+2=8$
$3x=6$	
$x=2$	
$3x+2=-8$	1 – for $3x+2=-8$
$3x=-10$	
$x=-\frac{10}{3}$	1 – mark for $x=-\frac{10}{3}$ and 2

c) (i) $\lim_{x\to 3^-} f(x) = 3 + 6 = 9$ $\lim_{x\to 3^+} f(x) = 3^2 = 9$ $\lim_{x\to 3} f(x) = 9$	1 – for determining the right hand and left hand limits 1 – stating $\lim_{x\to 3} f(x) = 9$
(ii) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ $f(3) = 9$ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$ Therefore, $f(x)$ is continuous at $x = 3$	1 – stating that the right hand limit equals the left hand limit 1 – determining $f(3)$ 1 – stating that $\lim_{x\to 3} f(x) = f(3)$ 1 – stating that f is continuous at $x = 3$
d) $f(x) = \cos 2x$ $f(x+h) = \cos(2x+2h)$ $f(x+h) - f(x) = \cos(2x+2h) - \cos 2x$ $= -2\sin\left(\frac{2x+2h+2x}{2}\right)\sin\left(\frac{2x+2h-2x}{2}\right)$ $\lim_{h\to 0}\frac{-2\sin(2x+h)\sin h}{h}$ $= \lim_{h\to 0} -2\sin(2x+h)\frac{\sin h}{h}$ $= -2\sin(2x+0)\times 1$ $= -2\sin 2x$	1 – expression for $f(x+h) - f(x)$ 1 – use of factor formula 1 – use of limit formula 1 – use of $\lim_{h\to 0}\frac{\sin x}{x} = 1$ 1 – evaluating limit at $h = 0$ 1 – C.A.O
2. $y = 16x + \frac{1}{x} = 16x + x^{-1}$ $\frac{dy}{dx} = 16 - x^{-2} = 16 - \frac{1}{x^2}$ when $x = 1$ $\frac{dy}{dx} = 16 - \frac{1}{1^2} = 15$ $y = 16(1) + \frac{1}{1} = 17$	1 – determining $\frac{dy}{dx}$ 1 – substituting $x = 1$ into $\frac{dy}{dx}$ 1 – evaluating $\frac{dy}{dx}$ when $x = 1$ 1 – evaluating y when $x = 1$

$y = mx + c$ $17 = 15(1) + c$ $2 = c$ $y = 15x + 2$	1 – correct use of $y = mx + c$ 1 – correct equation
3. (i) $y = \frac{x}{4+x^2}$ $\frac{dy}{dx} = \frac{1(4+x^2) - x(2x)}{(4+x^2)^2}$ $= \frac{4 + x^2 - 2x^2}{(4+x^2)^2}$ $= \frac{4 - x^2}{(4+x^2)^2}$	1 – correct numerator 1 – correct denominator 1 – simplification of the numerator
(ii) $\frac{4 - x^2}{(4+x^2)^2} = 0$ $4 - x^2 = 0$ $x^2 = 4$ $x = \pm 2$ $y = \frac{2}{4+2^2} = \frac{1}{4}$ $\left(2, \frac{1}{4}\right)$ $y = -\frac{2}{4+(-2)^2} = -\frac{1}{4}$ $\left(-2, -\frac{1}{4}\right)$	1 – use of $\frac{dy}{dx} = 0$ 1 – for $x = \pm 2$ 2 – 1 mark for each correct pair of coordinates
4. (i) $\tan 30° = \frac{h}{b}$ $b = \frac{3h}{\sqrt{3}}$ Volume $= \frac{1}{2}\left(\frac{6h}{\sqrt{3}}\right)(h)(40)$ $= \frac{120h^2}{\sqrt{3}}$ $= \frac{120\sqrt{3}h^2}{3}$ $= 40\sqrt{3}h^2$	1 – correct expression for base of triangle 1 – use of formula for volume of a prism 1 – simplification

(ii) $V = 40\sqrt{3}h^2$ $\frac{dV}{dh} = 80\sqrt{3}h\frac{dh}{dt}$ $200 = 80\sqrt{3}(5)\frac{dh}{dt}$ $\frac{200}{400\sqrt{3}} = \frac{dh}{dt}$ $\frac{1}{2\sqrt{3}} = \frac{dh}{dt}$	1 – determining $\frac{dV}{dh}$ 1 – use of parametric differentiation and substituting $h = 5$ 1 – correct value for $\frac{dh}{dt}$
5. $x = \sin\theta$ $\frac{dx}{d\theta} = \cos\theta$ $y = \cos 2\theta$ $\frac{dy}{d\theta} = -2\sin 2\theta$ $\frac{dy}{dx} = -\frac{2\sin 2\theta}{\cos\theta}$ $= -\frac{2(2\sin\theta\cos\theta)}{\cos\theta}$ $= -4\sin\theta$	1 – for $\frac{dx}{d\theta} = \cos\theta$ 1 – for $\frac{dy}{d\theta} = -2\sin 2\theta$ 1 – for $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$ 1 – for use of $\sin 2\theta = 2\sin\theta\cos\theta$
6. $u = \sin x + 1 \quad \rightarrow u - 1 = \sin x$ $\frac{du}{dx} = \cos x$ $\int (u-1)\cos x\,(u^5)\frac{du}{\cos x}$ $\int u^6 - u^5\,du$ $= \frac{u^7}{7} - \frac{u^6}{6} + c$ $= \frac{6u^7 - 7u^6}{42} + c$ $= \frac{u^6(6u-7)}{42} + c$	1 – differentiating u 1 – substituting $dx = \frac{du}{\cos x}$ 1 – substituting $u - 1 = \sin x$ 1 – simplifying integrand 1 – integrating correctly 1 – combining fraction 1 – correct factoring

$= \frac{(\sin x + 1)^6(6(\sin x + 1) - 7)}{42} + c$ $= \frac{(\sin x + 1)^6(6\sin x - 1)}{42} + c$	1 – resubstituting $u = \sin x + 1$
7. $x = \frac{12}{y^2} - 2$ $x^2 = \left(\frac{12}{y^2} - 2\right)\left(\frac{12}{y^2} - 2\right)$ $x^2 = \frac{144}{y^4} - \frac{48}{y^2} + 4$	1 – determining $x^2 = \frac{144}{y^4} - \frac{48}{y^2} + 4$
$\pi \int_1^2 (144y^{-4} - 48y^{-2} + 4)\ dy$	1 – substituting x^2 into correct formula for volume of revolution
$= \pi \left[\frac{144y^{-3}}{-3} - \frac{48y^{-1}}{-1} + 4y\right]_1^2$	1 – correct integration
$= \pi \left(\left[\frac{144}{-3(2)^3} + \frac{48}{2} + 4(2)\right] - \left[\frac{144}{-3(1)^3} + \frac{48}{1} + 4(1)\right]\right)$	1 – substitution of the upper limit 1 – substitution of the lower limit
$= 22\pi$	1 – C.A.O

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CARIBBEAN ADVANCED PROFICIENCY EXAMINATION

SCHOOL BASED ASSESSMENT

PURE MATHEMATICS

UNIT 1 – TEST 3 (PREVIEW)

Time: 1 Hour & 20 minutes

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8. (a) Determine

   (i) $\lim_{x \to 3} \dfrac{x^3 - 9x}{x - 3}$ [18]

   (ii) $\lim_{x \to 0} \dfrac{\sin 5x}{2x}$ $\left[\frac{5}{2}\right]$

   (b) Find the values of x for which $\frac{x^2+1}{|2x+3|-6}$ is discontinuous. $\left[-\frac{9}{2}, \frac{3}{2}\right]$

   (c) A function $f(x)$ is defined as

$$f(x) = \begin{cases} x + 2 & x \leq 2 \\ x^2 & x > 2 \end{cases}$$

   (iii) Find $\lim_{x \to 2} f(x)$. [4]

   (iv) Determine whether $f(x)$ is continuous at $x = 2$. Give a reason for your answer. [Yes]

   (d) Differentiate $f(x) = \sin 2x$ using first principles. [$2\cos 2x$]

Total 23 Marks

9. Given that $y = 8x + \frac{1}{x}$, determine the equation of the tangent to the curve at the point where $x = 1$.

[$y = 7x + 2$]

TOTAL 6 Marks

10. The curve C has equation $y = \frac{x}{1+x^2}$.

   (iii) Show that $\frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2}$

   (iv) Determine the coordinates of the stationary points on C. $\left[\left(1, \frac{1}{2}\right), \left(-1, -\frac{1}{2}\right)\right]$

Total 7 Marks

11.

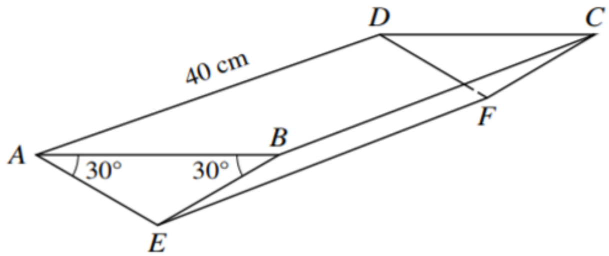


Fig. 1

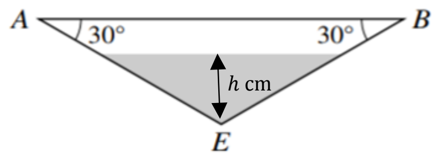


Fig. 2

Fig. 1 shows an open tank in the shape of a triangular prism. The vertical ends ABE and DCF are identical isosceles triangles, angle ABE = angle $BAE = 30°$. The length of AD is 40 cm. The tank is fixed in position with the open top $ABCD$ horizontal. Water is poured into the tank at a constant rate of 100 cm^3s^{-1}. The depth of water, t seconds after filling starts, is h cm (see Fig. 2).

(iii) Show that, when the depth of water in the tank is h cm, the volume, V cm^3, of water in the tank is given by $V = (40\sqrt{3})h^2$.

(iv) Find the rate at which h is increasing when $h = 4$. $\left[\frac{5\sqrt{3}}{48}\right]$

Total 6 Marks

12. The parametric equations of a curve are given by

$$x = \cos\theta\,, y = \sin 2\theta\,, \qquad 0 \le \theta \le 2\pi$$

find $\frac{dy}{dx}$. $\left[-\frac{2\cos 2\theta}{\sin\theta}\right]$

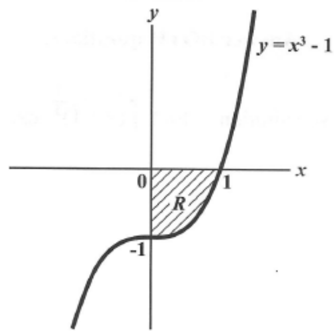
Total 4 Marks

13. Use the substitution $u = \sin x + 2$ to show that

$$\int \cos x\,(2+\sin x)^6 \; dx = \frac{(2+\sin x)^7}{7} + c$$

Total 8 Marks

14. The diagram below represents the finite region R which is enclosed by the curve $y = x^3 - 1$ and the lines $x = 0$ and $y = 0$.



Calculate the volume of the solid that results from rotating R about the $y -$ axis. $\left[\frac{3\pi}{5}\right]$

Total 6 Marks

15.

3

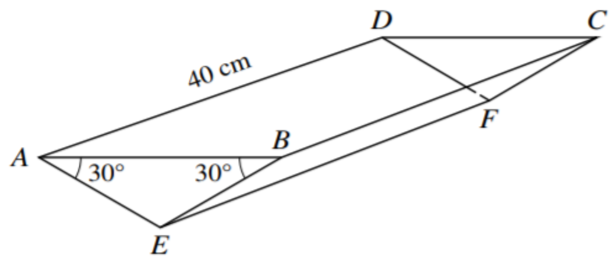


Fig. 1

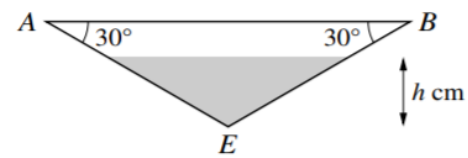


Fig. 2

Fig. 1 shows an open tank in the shape of a triangular prism. The vertical ends *ABE* and *DCF* are identical isosceles triangles. Angle *ABE* = angle *BAE* = 30°. The length of *AD* is 40 cm. The tank is fixed in position with the open top *ABCD* horizontal. Water is poured into the tank at a constant rate of 200 $\text{cm}^3\,\text{s}^{-1}$. The depth of water, *t* seconds after filling starts, is *h* cm (see Fig. 2).

(i) Show that, when the depth of water in the tank is *h* cm, the volume, $V\ \text{cm}^3$, of water in the tank is given by $V = (40\sqrt{3})h^2$. [3]

(ii) Find the rate at which *h* is increasing when *h* = 5. [3]

3	**(i)**	$\tan 60 = \frac{x}{h} \rightarrow x = h\tan 60$ $A = h \times x$ $V = 40\sqrt{(3h^2)}$	**B1** **M1** **A1** [3]	Any correct unsimplified length Correct method for area ag
	(ii)	$\frac{dV}{dh} = 80\sqrt{(3h)}$ If $h = 5$, $\frac{dh}{dt} = \frac{1}{2\sqrt{(3)}}$ or 0.289	**B1** **M1A1** [3]	B1 M1 (must be ÷, not ×).

16. a