HARRISON COLLEGE INTERNAL EXAMINATION MARCH 2022
CARIBBEAN ADVANCED PROFICIENCY EXAMINATION
SCHOOL BASED ASSESSMENT
PURE MATHEMATICS
UNIT I – TEST 1
Time: 1 hour and 20 minutes

NAME OF STUDENT: SBA TEST 1 MARK SCHEME
SCHOOL CODE: 030014
DATE: 14 MARCH 2022

This examination paper consists of 9 printed pages and 1 blank page for extra working.

The paper consists of **9** questions.

The maximum mark for this examination is **60**.

INSTRUCTIONS TO CANDIDATES

1. Write your name clearly in the space above.
2. Answer **EACH** question in the **SPACE PROVIDED**. **SHOW ALL WORKING.**
3. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided.
4. Number your questions **carefully and identically to those on the question paper**.
5. Unless otherwise stated in the question, any numerical answer that is not exact, **MUST** be written correct to three (3) significant figures

EXAMINATION MATERIALS ALLOWED

1. Mathematical formulae
2. Scientific calculator (non-programmable, non-graphical)

1) Given that p and q are propositions, use the algebra of propositions to simplify fully $\sim(p \vee q) \vee (\sim p \wedge q)$ [4]

Total: 4 marks

$= (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ (1)

$= \sim p \wedge (\sim q \vee q)$ (1)

$= \sim p \wedge (1)$ (1)

$= \sim p$ (1)

2) Prove that for all $x \in \boldsymbol{R}, y \in \boldsymbol{R}; x \geqslant 0, y \geqslant 0; x + y \geq -2\sqrt{xy}$ [3]

Total: 3 marks

$(\sqrt{x} + \sqrt{y})^2 \geqslant 0$ (1)

$(\sqrt{x})^2 + 2\sqrt{x}\sqrt{y} + (\sqrt{y})^2 \geqslant 0$ (1)

$x + 2\sqrt{xy} + y \geqslant 0$ (1)

$x + y \geqslant -2\sqrt{xy}$

3) Simplify <u>FULLY</u> $\frac{\sqrt{p}-\sqrt{2}}{\sqrt{p}+\sqrt{2}} - \frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$. [4]

Total: 4 marks

$$\frac{(\sqrt{p}-\sqrt{2})}{(\sqrt{p}+\sqrt{2})} - \frac{(\sqrt{p}+\sqrt{2})}{(\sqrt{p}-\sqrt{2})}$$

$$= \frac{(\sqrt{p}-\sqrt{2})^2 - (\sqrt{p}+\sqrt{2})^2}{(\sqrt{p}+\sqrt{2})(\sqrt{p}-\sqrt{2})} \quad 1 \; 1$$

$$= \frac{(p - 2\sqrt{p}\sqrt{2} + 2) - (p + 2\sqrt{p}\sqrt{2} + 2)}{(\sqrt{p})^2 - (\sqrt{2})^2} \quad 1$$

$$= \frac{-4\sqrt{p}\sqrt{2}}{p-2}$$

$$= -\frac{4\sqrt{2p}}{p-2} \quad 1$$

$$\text{OR} = \frac{4\sqrt{2p}}{2-p}$$

4) Prove by mathematical induction that $9^{2n} - 1$ is divisible by $8 \ \forall \ n \in N$. [6]

Total: 6 marks

Let P_n be the Proposition
"$9^{2n} - 1$ is divisible by $8 \ \forall \ n \in N$"

BASIC STEP: To show P_1 true
$n=1$: $P_1 \equiv 9^{2(1)} - 1$
$= 80$, which is divisible by 8
$\therefore P_1$ is TRUE

1

INDUCTIVE STEP: Assume P_k TRUE
i.e. $P_k \equiv 9^{2k} - 1$ is divisible by 8
$\forall \ k \in N, \ k > 1.$

1

We are required to show that
$P_{k+1} \equiv 9^{2(k+1)} - 1$ is divisible by 8

Let $9^{2k} - 1 = 8A$, $A \in N$

Now $P_{k+1} \equiv 9^{2k+2} - 1$ 1
$= 9^{2k} \cdot 9^2 - 1$ 1
$= (8A+1) \cdot 9^2 - 1$
$= 648A + 81 - 1$
$= 8(81A + 10)$
which is divisible by 8

1

CONCLUSION
$P_k \Rightarrow P_{k+1}$
ie. $P_1 \Rightarrow P_2$, $P_2 \Rightarrow P_3, \ldots, P_{n-1} \Rightarrow P_n$
Hence, by MI, P_n is true

1

5) The expression $2x^3 + ax^2 + bx + 1$ is exactly divisible by $2x - 1$ and $x + 1$.

(i) Determine the values of a and of b [5]

(ii) Find the third factor of the expression [2]

(ii) Hence, solve $2x^3 + ax^2 + bx + 1 = 0$. [4]

Total: 11 marks

(i) $P(x) = 2x^3 + ax^2 + bx + 1$

$P(\frac{1}{2}) = 0$

$\Rightarrow 2\cdot(\frac{1}{2})^3 + a(\frac{1}{2})^2 + b(\frac{1}{2}) + 1 = 0$ ①

$\frac{1}{4} + \frac{1}{4}a + \frac{1}{2}b + 1 = 0$

i.e. $1 + a + 2b + 4 = 0$

$a + 2b = -5$ eqⁿ ① ①

$P(\frac{-1}{1}) = 0$

$\Rightarrow 2(-1)^3 + a(-1)^2 + b(-1) + 1 = 0$

$a - b = 1$ eqⁿ ② ①

eqⁿ ① − eqⁿ ②: $3b = -6$

$b = -2$ ①

Sub. into eqⁿ ②

$a - (-2) = 1$

$a = -1$ ①

(ii) $(2x-1)(x+1) = 2x^2 + x - 1$

$$\begin{array}{r|l} & \quad\quad\quad x - 1 \\ 2x^2+x-1 & 2x^3 - x^2 - 2x + 1 \\ & 2x^3 + x^2 - x \\ \hline & \quad -2x^2 - x + 1 \\ & \quad -2x^2 - x + 1 \\ \hline & \quad\quad 0 \quad 0 \quad 0 \end{array}$$ ①

The Third Factor is $(x-1)$ ①

(iii) $2x^3 + ax^2 + bx + 1 = 0$

$2x^3 - x^2 - 2x + 1 = 0$

$\Rightarrow (2x-1)(x+1)(x-1) = 0$ ①

$\Rightarrow x = \frac{1}{2}, x = -1, x = 1$ ① + ① + ①

6) Solve for x

(a) $log_2(2x^2 + 3x + 5) = 3 + log_2(x + 1)$ [5]

(b) $2^{2x+1} - 15(2^x) = 8$ [5]

Total: 10 marks

(a) $\log_2(2x^2+3x+5) = 3 + \log_2(x+1)$

$\log_2(2x^2+3x+5) - \log_2(x+1) = 3$ (1)

$\log_2\left[\frac{(2x^2+3x+5)}{(x+1)}\right] = 3$ (1)

$\frac{(2x^2+3x+5)}{(x+1)} = 2^3$

$2x^2+3x+5 = 8x+8$

$2x^2-5x-3 = 0$ (1)

$(2x+1)(x-3) = 0$

$x = -\frac{1}{2}, x = 3$ (1) + (1)

(b) $2^{2x+1} - 15(2^x) = 8$

$2^{2x} \cdot 2^1 - 15(2^x) - 8 = 0$

$2(2^x)^2 - 15(2^x) - 8 = 0$ (1)

Let $y = 2^x$

$2y^2 - 15y - 8 = 0$

$(2y+1)(y-8) = 0$ (1)

$y = -\frac{1}{2}, y = 8$

$2^x = -\frac{1}{2}, 2^x = 8$ (1) + (1)

NOT VALID $\quad x = 3$ } (1) for BOTH

7) The number of mosquito larvae found in a pond initially was 4 000.

The number of larvae after t days, $N(t)$, was found to be directly proportional to $\left(\frac{3}{2}\right)^t$.

Calculate

(i) the number of larvae after 3 days [3]

(ii) the number of days for which the population is first expected to exceed 50 000. [4]

Total: 7 marks

(i) $N(t) \propto \left(\frac{3}{2}\right)^t$

i.e. $N(t) = k \cdot \left(\frac{3}{2}\right)^t$ ①

$N(0) = 4000$

$\Rightarrow k \cdot \left(\frac{3}{2}\right)^0 = 4000$

$k = 4000$ ①

$N(t) = 4000 \cdot \left(\frac{3}{2}\right)^t$

$N(3) = 4000 \cdot \left(\frac{3}{2}\right)^3$

$= 13,500$ ①

(ii) $N(t) > 50\,000$

$\Rightarrow 4000 \cdot \left(\frac{3}{2}\right)^t > 50\,000$ ①

$\left(\frac{3}{2}\right)^t > \frac{25}{2}$

$t \ln\left(\frac{3}{2}\right) > \ln\left(\frac{25}{2}\right)$ ①

$t > \dfrac{\ln\left(\frac{25}{2}\right)}{\ln\left(\frac{3}{2}\right)}$

$t > 6.23$ days ①

$t = 7$ days ①

8) Find the range of values of x for which $\left|1 - \frac{x}{3}\right| < 3$. [5]

Total: 5 marks

Either $+\left(1 - \frac{x}{3}\right) < 3$ exp. ① ①

OR $-\left(1 - \frac{x}{3}\right) < 3$ exp. ② ①

From exp. ①: $1 - 3 < \frac{x}{3}$

$-6 < x$ ①

From exp. ②: $1 - \frac{x}{3} > -3$

$1 + 3 > \frac{x}{3}$

$12 > x$ ①

Sol^{n}: $-6 < x < 12$ ①

Alternatively

$+\sqrt{\left(1 - \frac{x}{3}\right)} < 3$

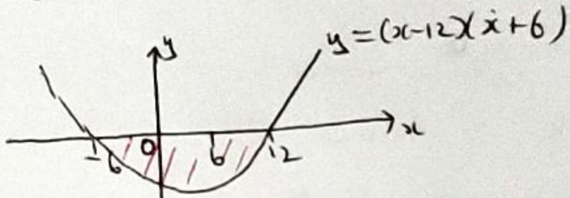
$\left(1 - \frac{x}{3}\right)^2 < 3^2$ ①

$1 - \frac{2x}{3} + \frac{x^2}{9} < 9$ ①

$9 - 6x + x^2 < 81$ ①

$x^2 - 6x - 72 < 0$ ①

$(x - 12)(x + 6) < 0$



Sol^{n}: $-6 < x < 12$ ①

9) If $\boldsymbol{\alpha}, \boldsymbol{\beta}$ **and** $\boldsymbol{\gamma}$ are the roots of the equation $3x^3 - 4x^2 - 5x + 2 = 0$

(a) Find the values of

(i) $\alpha + \beta + \gamma$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$

(iii) $\alpha\beta\gamma$ [3]

(b) Hence, or otherwise, find the equation with roots $\alpha - 1, \beta - 1$ and $\gamma - 1$. [7]

Total: 10 marks

(a) (i) $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{4}{3}$ ①

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = -\frac{5}{3}$ ①

(iii) $\alpha\beta\gamma = -\frac{d}{a} = -\frac{2}{3}$ ①

(b) Sum: $(\alpha - 1) + (\beta - 1) + (\gamma - 1)$

$= \alpha + \beta + \gamma - 3$ ①

$= \frac{4}{3} - 3$

$= -\frac{5}{3}$ ①

Sum Pairwise: $(\alpha-1)(\beta-1)+(\alpha-1)(\gamma-1)+(\beta-1)(\gamma-1)$

$=(\alpha\beta-\alpha-\beta+1)+(\alpha\gamma-\alpha-\gamma+1)+(\beta\gamma-\beta-\gamma+1)$

$=\alpha\beta+\alpha\gamma+\beta\gamma-2(\alpha+\beta+\gamma)+3$ ①

$=-\frac{5}{3}-2\left(\frac{4}{3}\right)+3$

$=-\frac{4}{3}$ ①

Product: $(\alpha-1)(\beta-1)(\gamma-1)$

$=(\alpha-1)(\beta\gamma-\beta-\gamma+1)$

$=\alpha\beta\gamma-\alpha\beta-\alpha\gamma+\alpha-\beta\gamma+\beta+\gamma-1$

$=\alpha\beta\gamma-(\alpha\beta+\alpha\gamma+\beta\gamma)+(\alpha+\beta+\gamma)-1$ ①

$=-\frac{2}{3}-\left(-\frac{5}{3}\right)+\frac{4}{3}-1$

$=\frac{4}{3}$ ①

Req'd Eq^n is

$$\left.\begin{aligned}&x^3-\left(-\tfrac{5}{3}\right)x^2+\left(-\tfrac{4}{3}\right)x-\tfrac{4}{3}=0\\&3x^3+5x^2-4x-4=0\end{aligned}\right\}$$ ①

Alternatively Part(b)

$3x^3-4x^2-5x+2=0$

$x=\alpha,\beta,\gamma$

$X=x-1$ ①

$X+1=x$ ①

$3(X+1)^3-4(X+1)^2-5(X+1)+2=0$ ①

$3(X^3+3X^2+3X+1)-4(X^2+2X+1)-5X-5+2=0$ ①

$3X^3+9X^2+9X+3-4X^2-8X-4-5X-3=0$ ①

$3X^3+5X^2-4X-4=0$ ①

i.e. $3x^3+5x^2-4x-4=0$ ①

End of Examination