CAPE UNIT II – Test 2, 2020 Preview

1. The population of a town is 50 000 at the end of Year 1.

   A model predicts that the population of the town will increase by 2.5% each year.

   a) Determine the predicted population at the end of Year 2. [1]

   [51 250]

The model predicts that Year N will be the first year in which the population of the town exceeds 500 000.

   b) Show that $(N-1)\log 1.025 > 1$ [4]

   c) Find the value of N. [2]

   [95]

   At the end of each year, a donation of 50 cents will be made to a charity fund on behalf of each member of the population. Assuming the population model,

   d) Find the total amount that will be given to the charity fund for the 20 years from the end of Year 1, to the end of year 20, giving your answer to the nearest \$1 000. [3]

   [\$639 000]

2. Show that

$$\frac{r+1}{r+2} - \frac{r}{r+1} \equiv \frac{1}{(r+1)(r+2)}$$

[2]

ii) Hence, using the method of differences, find an expression in terms of n, for

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{(n+1)(n+2)}.$$

[5]

[$\frac{n}{2(n+2)}$]

iii) Hence show that

$$\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+2)} = \frac{1}{2}$$

[1]

3. A series of positive integers u_1, u_2, u_3 ... is defined by

$$u_1 = 2 \; and \; u_{n+1} = 5u_n - 4, for \; n \geq 1.$$

Prove by mathematical induction that $\boldsymbol{u_n} = (\mathbf{5^{n-1}}) + \mathbf{1}$, for $n \geq 1$. [8]

4.

$$f(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

a) Show that the equation $f(x) = 0$ has a root in the interval $[2, 2.5]$ [3]

b) Using $x_1 = 2.25$ as a first approximation to the real root α, apply the Newton-Raphson procedure once to find a second approximation to α, giving your answer to 3 significant figures. [5]

[2.17]

c) Starting with the interval $[2, \; 2.5]$, use the interval bisection method twice to obtain an approximation, giving your answer to 3 significant figures. [3]

[2.13]

5. a) Use the binomial theorem to expand

$$\sqrt[3]{(8-9x)}, |x| < \frac{8}{9},$$

in ascending powers of x, up to and including the term in x^3, giving each term as a simplified fraction.

$[2 - \frac{3}{4}x - \frac{9}{32}x^2 - \frac{45}{256}x^3 \ldots \;]$ [5]

b) Use your answer with a suitable value of x, to obtain an approximation to $\sqrt[3]{7100}$. Give your answer to 4 decimal places. [19.2201] [3]

6. a) Given that $y^2 = \sec x + \tan x, for -\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0$, show that

i) $\frac{dy}{dx} = \frac{1}{2} y \sec x$ [3]

ii) $\frac{d^2y}{dx^2} = \frac{1}{4} y \sec x (2tan\ x + \sec x)$ [3]

b) Given that x is small and terms in x^3 and higher powers of x may be neglected, obtain the Maclaurin expansion for the function y. [2]

[$1 + \frac{1}{2}x + \frac{1}{8}x^2$]