CAPE 2019

1. (a) Let $4x^2 + 3xy^2 + 7x + 3y = 0$.

(i) Use implicit differentiation to show that

$$\frac{dy}{dx} = -\frac{8x + 3y^2 + 7}{3(1 + 2xy)}$$

[5]

(ii) Show that for $f(x, y) = 4x^2 + 3xy^2 + 7x + 3y$,

$$6\frac{\partial f(x,y)}{\partial y} - 10 = \left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)\left(\frac{\partial^2 f(x,y)}{\partial y \partial x}\right) + \frac{\partial^2 f(x,y)}{\partial x^2}$$

[5]

i) $4x^2 + 3xy^2 + 7x + 3y = 0$

$8x + 3y^2 + 3x\left[2y\frac{dy}{dx}\right] + 7 + 3\frac{dy}{dx} = 0$

$(6xy + 3)\frac{dy}{dx} = -(8x + 3y^2 + 7)$

$\frac{dy}{dx} = -\frac{8x + 3y^2 + 7}{3 + 6xy}$

$= -\frac{8x + 3y^2 + 7}{3(1 + 2xy)}$

ii) $f(x, y) = 4x^2 + 3xy^2 + 7x + 3y$

$\frac{\partial f}{\partial y} = 0 + 6xy + 3 = 3 + 6xy$

$\frac{\partial^2 f}{\partial y^2} = 6x$

$\frac{\partial^2 f}{\partial y \partial x} = 6y$

$\frac{\partial f}{\partial x} = 8x + 3y^2 + 7$

$\frac{\partial^2 f}{\partial x^2} = 8$

$6\frac{\partial f}{\partial y} - 10 = 6(3 + 6xy) - 10$

$= 18 + 36xy - 10$

$= 8 + 36xy$

$\left(\frac{\partial^2 f}{\partial y^2}\right)\left(\frac{\partial^2 f}{\partial y \partial x}\right) + \frac{\partial^2 f}{\partial x^2} = (6x)(6y) + 8$

$= 36xy + 8$

(b) Use de Moivre's theorem to prove that $\sin 5x = 16\sin^5 x - 20\sin^3 x + 5\sin x$. [6]

$$(\cos x + i\sin x)^5 = \binom{5}{0}\cos^5 x + \binom{5}{1}\cos^4 x(i\sin x) + \binom{5}{2}\cos^3 x(i\sin x)^2 + \binom{5}{3}\cos^2 x(i\sin x)^3$$
$$+ \binom{5}{4}\cos x(i\sin x)^4 + \binom{5}{5}(i\sin x)^5$$
$$= \cos^5 x + 5i\cos^4 x\sin x - 10\cos^3 x\sin^2 x - 10i\cos^2 x\sin^3 x$$
$$+ 5\cos x\sin^4 x + i\sin^5 x$$

$$\sin 5x = 5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x$$
$$= 5\cos^2 x\cos^2 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x$$
$$= 5(1-\sin^2 x)(1-\sin^2 x)\sin x - 10(1-\sin^2 x)\sin^3 x + \sin^5 x$$
$$= 5(1-2\sin^2 x+\sin^4 x)\sin x - 10(\sin^3 x - \sin^5 x) + \sin^5 x$$
$$= (5-10\sin^2 x + 5\sin^4 x)\sin x - 10\sin^3 x + 10\sin^5 x + \sin^5 x$$
$$= 5\sin x - 10\sin^3 x + 5\sin^5 x - 10\sin^3 x + 10\sin^5 x + \sin^5 x$$
$$= 16\sin^5 x - 20\sin^3 x + 5\sin x$$

(c) (i) Write the complex number $z = \left(-1+\sqrt{3}i\right)^7$ in the form $re^{i\theta}$, where $r|z|$ and $\theta = \arg z$. [3]

(ii) Hence, prove that $\left(-1+\sqrt{3}\,i\right)^7 = 64\left(-1+\sqrt{3}\,i\right)$. [6]

i) Let $z_1 = -1+\sqrt{3}i$

$r_1 = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$

$\arg z_1 = \pi - \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{2\pi}{3}$

$z_1 = 2e^{\frac{2\pi}{3}i}$

$z = \left(2e^{\frac{2\pi}{3}i}\right)^7$

$= 128e^{\frac{14\pi}{3}i}$

$= 128e^{\frac{2\pi}{3}i}$

ii) $r(\cos\theta + i\sin\theta)$

$= 128\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

$= 128\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= 64(-1+\sqrt{3}i)$

2. (a) Let $F_n(x) = \int (\ln x)^n \; dx$

(i) Show that $F_n(x) = x(\ln x)^n - nF_{n-1}(x)$. [3]

(ii) Hence, or otherwise, show that $F_3(2) - F_3(1) = 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6$. [7]

i) $\int 1 (\ln x)^n \, dx$

$u = (\ln x)^n \qquad v = 1$

$du = n(\ln x)^{n-1}\left(\frac{1}{x}\right) \qquad dv = x$

$F_n(x) = x(\ln x)^n - \int x\left[n(\ln x)^{n-1}\left(\frac{1}{x}\right)\right] dx$

$= x(\ln x)^n - n\int (\ln x)^{n-1} dx$

$= x(\ln x)^n - n F_{n-1}(x)$

ii) $F_3(2) = 2(\ln 2)^3 - 3F_2(2)$

$= 2(\ln 2)^3 - 3\left[2(\ln 2)^2 - 2F_1(2)\right]$

$= 2(\ln 2)^3 - 6(\ln 2)^2 + 6F_1(2)$

$= 2(\ln 2)^3 - 6(\ln 2)^2 + 6\left[2(\ln 2)^1 - 1F_0(2)\right]$

$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6x$

$F_3(1) = 1(\ln 1)^3 - 6(\ln 1)^2 + 12\ln 1 - 6x$

$= -6x$

$= -6$

$F_3(2) - F_3(1) = 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6(2) - (-6)$

$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6$

(b) (i) By expressing $\frac{y^2+2y+1}{y^4+2y^2+1}$ as partial fractions, show that

$$\frac{y^2+2y+1}{y^4+2y^2+1} = \frac{1}{y^2+1} + \frac{2y}{(y^2+1)^2}$$

[7]

(ii) Hence, or otherwise, evaluate

$$\int \frac{y^2+2y+1}{y^4+2y^2+1}\, dy$$

[8]

i) $\dfrac{y^2+2y+1}{y^4+2y^2+1} = \dfrac{y^2+2y+1}{(y^2+1)^2} = \dfrac{Ay+B}{y^2+1} + \dfrac{Cy+D}{(y^2+1)^2}$

$y^2+2y+1 = (Ay+B)(y^2+1) + Cy+D$

$= Ay^3 + By^2 + Ay + B + Cy + D$

$= Ay^3 + By^2 + (A+C)y + (B+D)$

Equating y^3 terms

$A = 0$

Equating y^2 terms

$B = 1$

Equating y terms

$A + C = 2$

$C = 2$

Equating constants

$B + D = 1$

$D = 0$

$\dfrac{y^2+2y+1}{y^4+2y^2+1} = \dfrac{1}{y^2+1} + \dfrac{2y}{(y^2+1)^2}$

ii) $\displaystyle\int \frac{y^2+2y+1}{y^4+2y^2+1}\, dy = \int \frac{1}{y^2+1}\, dy + \int \frac{2y}{(y^2+1)^2}\, dy$

$= \tan^{-1}y + \int 2y(y^2+1)^{-2}\, dy$

$= \tan^{-1}y + \dfrac{(y^2+1)^{-2+1}}{-2+1} + c$

$= \tan^{-1}y - (y^2+1)^{-1} + c$

$= \tan^{-1}y - \dfrac{1}{y^2+1} + c$

3. (a) Determine the coefficient of the term in x^3 in the binomial expansion of $(3x + 2)^5$. [3]

$\binom{5}{2} (3x)^3 (2)^2$

$= 10 \times 27 \times 4 x^3$

$= 1080$

(b) (i) Show that the binomial expansion of $(1+x)^{\frac{1}{4}} + (1-x)^{\frac{1}{4}}$ up to the term in x^2 is $2 - \frac{3}{16}x^2$. [4]

(ii) Hence, by letting $x = \frac{1}{16}$, compute an approximation of $\sqrt[4]{17} + \sqrt[4]{15}$, correct to 4 decimal places. [3]

i) $(1+x)^{1/4} = 1 + \frac{1}{4}x + \frac{\frac{1}{4}(\frac{1}{4}-1)}{2!}x^2$

$= 1 + \frac{1}{4}x - \frac{3}{32}x^2$

$(1-x)^{1/4} = 1 + \frac{1}{4}(-x) + \frac{\frac{1}{4}(\frac{1}{4}-1)}{2!}(-x)^2$

$= 1 - \frac{1}{4}x - \frac{3}{32}x^2$

$(1+x)^{1/4} + (1-x)^{1/4} = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + 1 - \frac{1}{4}x - \frac{3}{32}x^2$

$= 2 - \frac{3}{16}x^2$

ii) $\left(1+\frac{1}{16}\right)^{1/4} + \left(1-\frac{1}{16}\right)^{1/4} = \left(\frac{17}{16}\right)^{1/4} + \left(\frac{15}{16}\right)^{1/4}$

$= \frac{17^{1/4}}{2} + \frac{15^{1/4}}{2}$

$2 - \frac{3}{16}x^2$ $\quad = \frac{\sqrt[4]{17} + \sqrt[4]{15}}{2}$

$2\left(2 - \frac{3}{16}x^2\right) = \sqrt[4]{17} + \sqrt[4]{15}$

$2\left(2 - \frac{3}{16}\left(\frac{1}{16}\right)^2\right) = \sqrt[4]{17} + \sqrt[4]{15}$

$3.9985 = \sqrt[4]{17} + \sqrt[4]{15}$

(c) The function $h(x) = x^3 + x - 1$ is defined on the interval [0, 1].

(i) Show that $h(x) = 0$ has a root on the interval [0, 1]. [3]

(ii) Use the iteration $x_{n+1} = \frac{1}{x_n^2+1}$ with initial estimate $x_1 = 0.7$ to estimate the root of $h(x) = 0$, correct to 2 decimal places. [6]

i) $h(x) = x^3 + x - 1$

$h(0) = 0^3 + 0 - 1 = -1$

$h(1) = 1^3 + 1 - 1 = 1$

$h(x)$ is continuous on the interval $[0,1]$

$h(0) \times h(1) < 0$

By the Intermediate Value Theorem there must be some $c \in [0,1]$ such that $h(c) = 0$. Therefore, there is a root between 0 and 1.

ii) $x_{n+1} = \frac{1}{x_n^2+1}$

$x_1 = 0.7$

$x_2 = \frac{1}{x_1^2+1} = 0.6711$

$x_3 = 0.6895$

$x_4 = 0.6778$

$x_5 = 0.6852$

$x_6 = 0.6805$

$x_7 = 0.6835$

Root is 0.68 to 2 decimal places

(d) Use the Newton – Raphson method with initial estimate $x_1 = 5.5$ to approximate the root of $g(x) = \sin 3x$ in the interval [5, 6], correct to 2 decimal places. [6]

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$g(x) = \sin 3x$$

$$g'(x) = 3\cos 3x$$

$$x_{n+1} = x_n - \frac{\sin 3x}{3\cos 3x}$$

$$x_2 = 5.5 - \frac{\sin 3(5.5)}{3\cos 3(5.5)} = 5.16$$

$$x_3 = 5.24$$

$$x_4 = 5.24$$

4. (a) A function is defined as $g(x) = x\sin\left(\frac{x}{2}\right)$.

(i) Obtain the Maclaurin series expansion for g up to the term in x^4. [8]

(ii) Hence, estimate $g(2)$. [2]

i) $g(x) = x\sin\left(\frac{x}{2}\right) \rightarrow g(0) = 0$

$g'(x) = 1\sin\left(\frac{x}{2}\right) + x\cos\left(\frac{x}{2}\right)\times\frac{1}{2}$

$= \sin\left(\frac{x}{2}\right) + \frac{x}{2}\cos\left(\frac{x}{2}\right) \rightarrow g'(0) = 0$

$g''(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right) + \frac{1}{2}\cos\left(\frac{x}{2}\right) + \frac{x}{2}\left[-\sin\left(\frac{x}{2}\right)\times\frac{1}{2}\right]$

$= \cos\left(\frac{x}{2}\right) - \frac{x}{4}\sin\left(\frac{x}{2}\right) \rightarrow g''(0) = 1$

$g'''(x) = -\frac{1}{2}\sin\left(\frac{x}{2}\right) - \left[\frac{1}{4}\sin\left(\frac{x}{2}\right) + \frac{x}{4}\left[\frac{1}{2}\cos\left(\frac{x}{2}\right)\right]\right]$

$= -\frac{1}{2}\sin\left(\frac{x}{2}\right) - \frac{1}{4}\sin\left(\frac{x}{2}\right) - \frac{x}{8}\cos\left(\frac{x}{2}\right)$

$= -\frac{3}{4}\sin\left(\frac{x}{2}\right) - \frac{x}{8}\cos\left(\frac{x}{2}\right) \rightarrow g'''(0) = 0$

$g''''(x) = -\frac{3}{4}\left(\frac{1}{2}\cos\left(\frac{x}{2}\right)\right) - \left[\frac{1}{8}\cos\left(\frac{x}{2}\right) + \frac{x}{8}\left(-\frac{1}{2}\sin\left(\frac{x}{2}\right)\right)\right)$

$= -\frac{3}{8}\cos\left(\frac{x}{2}\right) - \frac{1}{8}\cos\left(\frac{x}{2}\right) + \frac{x}{16}\sin\left(\frac{x}{2}\right)$

$= -\frac{1}{2}\cos\left(\frac{x}{2}\right) + \frac{x}{16}\sin\left(\frac{x}{2}\right) \rightarrow g''''(0) = -\frac{1}{2}$

$g(x) = \frac{x^2}{2!}(1) + \frac{x^4}{4!}\times\left(-\frac{1}{2}\right)$

$= \frac{x^2}{2} - \frac{x^4}{48}$

ii) $g(2) = \frac{2^2}{2} - \frac{2^4}{48}$

$= \frac{5}{3}$

(b) A series is given as $2+\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\cdots$

(i) Express the nth partial sum S_n of the series using sigma notation. [2]

(ii) Hence, calculate $S_{20}-S_{18}$, [1]

(iii) Given that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, show that S_n diverges. [4]

i) $S_n = \sum_{r=1}^{n} \frac{r+1}{r^2}$

ii) $S_{20} - S_{18} = \frac{19+1}{19^2} + \frac{20+1}{20^2} \approx 0.1079$

iii) $S_n = \sum_{n=1}^{\infty} \frac{n+1}{n^2} = \sum_{n=1}^{\infty} \frac{n}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2}$

$= \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$

$S_M = \sum_{M=1}^{\infty} \frac{1}{m}$

S_{2^0} $S_1 = 1$

S_{2^1} $S_2 = 1 + \frac{1}{2}$

S_{2^2} $S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$

S_{2^3} $S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} = 1 + \frac{2}{2} + \frac{1}{2} = 1 + \frac{3}{2}$

S_{2^4} $S_{16} = 1 + \frac{4}{2}$

$S_M = S_{2^n} = 1 + \frac{n}{2}$

As $n \to \infty$, $S_m \to \infty$

$\therefore S_M$ is divergent

$\therefore S_n = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$ diverges

(c) Use the method of induction to prove that

$$\sum_{r=1}^{n} r(r-1) = \frac{n(n^2-1)}{3}$$

[8]

$P_n : \sum_{r=1}^{n} r(r-1) = \frac{n(n^2-1)}{3}$

$P_1 : \quad 1(1-1) = \frac{1(1^2-1)}{3}$

$0 = 0$

$\therefore P_1$ is true

Assume P_n is true for all $n = k$

$P_k : \sum_{r=1}^{k} r(r-1) = \frac{k(k^2-1)}{3}$

$P_{k+1} : \sum_{r=1}^{k+1} r(r-1) = \frac{(k+1)((k+1)^2-1)}{3} = \frac{(k+1)(k^2+2k+1-1)}{3}$

$= \frac{(k+1)(k^2+2k)}{3}$

Now, $P_{k+1} = P_k + (k+1)\text{term}$

$= \frac{k(k^2-1)}{3} + (k+1)(k+1-1)$

$= \frac{k(k^2-1)}{3} + \frac{3k(k+1)}{3}$

$= \frac{k(k^2-1) + 3k(k+1)}{3}$

$= \frac{k(k+1)(k-1) + 3k(k+1)}{3}$

$= \frac{(k+1)[k(k-1) + 3k]}{3}$

$= \frac{(k+1)(k^2 - k + 3k)}{3}$

$= \frac{(k+1)(k^2+2k)}{3}$

$\therefore P_{k+1}$ is true wheneve P_k is true.

Hence by Mathematical Induction $\sum_{r=1}^{n} r(r-1) = \frac{n(n^2-1)}{3}$.

5. (a) (i) How many numbers made up of 5 digits can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if each number contains exactly one even digit and no digit is repeated? [4]

(ii) Determine the probability that the number formed in (a) (i) is less than 30 000. [4]

i) $\underline{\ \ }\ \underline{5}\ \underline{\ \ }\ \underline{4}\ \underline{\ \ }\ \underline{3}\ \underline{\ \ }\ \underline{2}\ \underline{\ \ }$

$= 4 \times 5 \times 4 \times 3 \times 2 \times 5$

OR

$= 4 \times {}^5P_4 \times 5$

$= 2400$

ii) Number begins with 2

$\underline{1}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}$

$= 1 \times 5 \times 4 \times 3 \times 2$

$= 1 \times {}^5P_4$

$= 120$

Number begins with 1

$\underline{1}\ \underline{\ \ }\ \underline{4}\ \underline{\ \ }\ \underline{3}\ \underline{\ \ }\ \underline{2}\ \underline{\ \ }$

$= 1 \times 4 \times 3 \times 2 \times 4 \times 4$

$= 384$

Total number of numbers less than 30 000 is 504

Probability $= \frac{504}{2400} = 0{\cdot}21$

(b) A and B are two matrices given below.

$$A = \begin{pmatrix} 2 & x & -1 \\ 3 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 2 & 1 & 2 \end{pmatrix}$$

(i) Determine the value of x for which A^{-1} does NOT exist. [4]

A^{-1} does not exist when $|A| = 0$

$$|A| = 2\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - x\begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= 2(0\times0 - 2\times1) - x(3\times0 - 2\times2) - 1(3\times1 - 0\times2)$$

$$= 2(-2) - x(-4) - 1(3)$$

$$= -4 + 4x - 3$$

$$= 4x - 7$$

If A^{-1} does not exist then

$$4x - 7 = 0$$

$$4x = 7$$

$$x = \frac{7}{4}$$

(ii) Given that $\det(AB) = -10$, show that $x = 2$. [4]

$$|AB| = -10$$
$$|A|\,|B| = -10$$
$$(4x-7)\,|B| = -10$$
$$|B| = 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}$$
$$= 1(3\times2 - 4\times1) - 2(2\times2 - 4\times2) + 5(2\times1 - 3\times2)$$
$$= 1(2) - 2(-4) + 5(-4)$$
$$= 2 + 8 - 20$$
$$= -10$$
$$(4x-7)(-10) = -10 \qquad 4x = 8$$
$$4x - 7 = 1 \longrightarrow x = 2$$

(iii) Hence, obtain A^{-1}. [4]

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 3 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \qquad |A| = 4x - 7 = 4(2) - 7 = 1$$

$$X = \begin{pmatrix} +\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} \end{pmatrix}$$

$$X = \begin{pmatrix} -2 & 4 & 3 \\ -1 & 2 & 2 \\ 4 & -7 & -6 \end{pmatrix}$$

$$X^{T} = \begin{pmatrix} -2 & -1 & 4 \\ 4 & 2 & -7 \\ 3 & 2 & -6 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -2 & -1 & 4 \\ 4 & 2 & -7 \\ 3 & 2 & -6 \end{pmatrix}$$

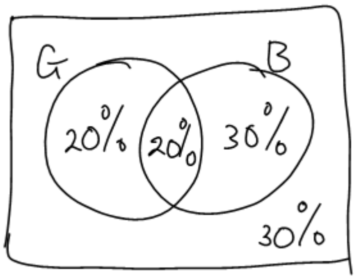
(c) In an experiment, individuals were asked to select from two available colours, green and blue. The individuals selected one colour, two colours or no colour.

70% of the individuals selected at least one colour and 600 individuals selected no colour.

(i) Given that 40% of the individuals selected green and 50% selected blue, calculate the probability that an individual selected BOTH colours. [3]

(ii) Determine the total number of individuals who participated in the experiment. [2]

i)



P(Both) = 20%

ii) 30% = 600

1% = 20

100% = 2000 persons

6. (a) A differential equation is given as

$$x\frac{dy}{dx} + y = 2\sin x$$

(i) Show that the general solution of the differential equation is $y = \frac{c}{x} - \frac{2}{x}\cos x$, where c is a constant. [5]

(ii) Hence, determine the particular solution of the differential equation that satisfies the condition $y = 2$ when $x = \pi$. [3]

i) $x\frac{dy}{dx} + y = 2\sin x$

$\int\left(x\frac{dy}{dx} + y\right)dx = \int 2\sin x\,dx$

$yx = -2\cos x + c$

$y = \frac{c}{x} - \frac{2}{x}\cos x$

ii) $2 = \frac{c}{\pi} - \frac{2}{\pi}\cos(\pi)$

$2 = \frac{c}{\pi} + \frac{2}{\pi}$

$2 = \frac{c+2}{\pi}$

$2\pi = c + 2$

$2\pi - 2 = c$

$y = \frac{2\pi - 2}{x} - \frac{2}{x}\cos x$

(b) Show that the general solution of the differential equation $\frac{dy}{dx} = \frac{xy-y}{x^2-4}$ is $y = k\sqrt[4]{(x-2)(x+2)^3}$ where k is a constant. [7]

$$\frac{dy}{dx} = \frac{xy-y}{x^2-4}$$

$$\frac{dy}{dx} = \frac{y(x-1)}{x^2-4}$$

$$\frac{1}{y}\,dy = \frac{x-1}{x^2-4}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{x-1}{x^2-4}\,dx$$

$$\ln y = \int \frac{3}{4(x+2)}\,dx + \int \frac{1}{4(x-2)}\,dx$$

$$= \frac{3}{4}\int \frac{1}{x+2}\,dx + \frac{1}{4}\int \frac{1}{x-2}\,dx$$

$$\ln y = \frac{3}{4}\ln(x+2) + \frac{1}{4}\ln(x-2) + c$$

$$= \ln(x+2)^{3/4} + \ln(x-2)^{1/4} + c$$

$$= \ln (x+2)^{3/4}(x-2)^{1/4} + c$$

$$\ln y = \ln \sqrt[4]{(x-2)^1(x+2)^3} + c$$

$$e^{\ln y} = e^{\ln \sqrt[4]{(x-2)(x+2)^3} + c}$$

$$y = e^c e^{\ln \sqrt[4]{(x-2)(x+2)^3}}$$

$$y = e^c \sqrt[4]{(x-2)(x+2)^3}$$

$$y = k\sqrt[4]{(x-2)(x+2)^3} \quad \text{when } k = e^c$$

$$\frac{x-1}{x^2-4} = \frac{x-1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$x-1 = A(x-2) + B(x+2)$$

when $x = 2$

$$2-1 = 4B$$

$$1 = 4B$$

$$\frac{1}{4} = B$$

when $x = -2$

$$-2-1 = -4A$$

$$-3 = -4A$$

$$\frac{3}{4} = A$$

(c) Solve the boundary – value problem $y'' - y' - 2y = 0$, given that when $x = -1, y = 1$ and when $x = 1, y = 0$. [10]

$y'' - y' - 2y = 0$

Auxiliary equation

$u^2 - u - 2 = 0$

$(u+1)(u-2) = 0$

$u = -1, 2$

$y = Ae^{-x} + Be^{2x}$

when $x = -1$, $y = 1$

$1 = Ae^{-(-1)} + Be^{2(-1)}$

$1 = Ae^{1} + Be^{-2}$ (1)

when $x = 1$, $y = 0$

$0 = Ae^{-1} + Be^{2(1)}$

$0 = Ae^{-1} + Be^{2}$ (2)

Solving (1) and (2) simultaneously

$1 = Ae^{1} + \frac{B}{e^2}$

$0 = \frac{A}{e} + Be^2 \quad \times e^2$

$1 = Ae^{1} + \frac{B}{e^2}$

$-$

$0 = Ae^{1} + Be^{4}$

$1 = \frac{B}{e^2} - Be^4$

$1 = B\left(\frac{1}{e^2} - e^4\right)$

$1 = B\left(\frac{1 - e^6}{e^2}\right)$

$\frac{e^2}{1 - e^6} = B$

Sub $B = \frac{e^2}{1 - e^6}$ into (1)

$1 = Ae^{1} + \frac{B}{e^2}$

$1 = Ae^{1} + \left(\frac{e^2}{1 - e^6}\right)\left(\frac{1}{e^2}\right)$

$$1 = Ae^{1} + \frac{1}{1-e^{6}}$$

$$1 - \frac{1}{1-e^{6}} = Ae^{1}$$

$$\frac{1-e^{6}-1}{1-e^{6}} = Ae^{1}$$

$$\frac{-e^{6}}{1-e^{6}} = Ae^{1}$$

$$\frac{-e^{5}}{1-e^{6}} = A$$

$$y = Ae^{-x} + Be^{2x}$$

$$y = \frac{-e^{5}}{1-e^{6}} e^{-x} + \frac{e^{2}}{1-e^{6}} e^{2x}$$