CAPE UNIT 2 (2018)

QUESTION 1

(a) (i)

Determine $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in order to use the result $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$x = \frac{t}{1+t}$$

$$\frac{dx}{dt} = \frac{1(1+t) - t(1)}{(1+t)^2} = \frac{t}{(1+t)^2}$$

$$y = \frac{t^3}{1+t}$$

$$\frac{dy}{dt} = \frac{3t^2(1+t) - t^3(1)}{(1+t)^2} = \frac{3t^2 + 3t^3 - t^3}{(1+t)^2} = \frac{3t^2 + 2t^3}{(1+t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{3t^2 + 2t^3}{(1+t)^2} \times \frac{(1+t)^2}{1} = 3t^2 + 2t^3$$

Determine the corresponding value of t when $x = \frac{1}{2}$ or $\left(y = \frac{1}{2}\right)$

$$\frac{1}{2} = \frac{t}{1+t}$$

$$1 + t = 2t$$

$$1 = t$$

$$\therefore \frac{dy}{dx} = 3(1)^2 + 2(1)^3 = 5$$

(a) (ii)

Equation of tangent

$$y = mx + c$$

$$\frac{1}{2} = 5\left(\frac{1}{2}\right) + c$$

$$-2 = c$$

$y = 5x - 2$

Therefore y intercept is -2.

At x intercept $y = 0$

$0 = 5x - 2$

$\frac{2}{5} = x \quad \rightarrow \quad x$ intercept

(b)

$f(x, y) = \sin(kx)\sin(aky)$

$\frac{\partial f}{\partial x} = k\sin(aky)\cos(kx)$

$\frac{\partial^2 f}{\partial x \partial y} = ak^2\cos(kx)\cos(aky)$

(c)

When $n = 5$

$\cos 5\theta + i\sin 5\theta$

$= (\cos\theta + i\sin\theta)^5$

$= 1\cos^5\theta + 5\cos^4\theta\,(i\sin\theta) + 10\cos^3\theta\,(i\sin\theta)^2 + 10\cos^2\theta\,(i\sin\theta)^3 + 5\cos\theta\,(i\sin\theta)^4 + 1(i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating Imaginary Parts

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

(d)

(i) $z = 1 - i$

$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\arg z = -\tan^{-1}\left(\frac{1}{1}\right) = -\frac{\pi}{4}$

$z = \sqrt{2}e^{-\frac{\pi}{4}i}$

(ii) $(1 - i)^9 = \left(\sqrt{2}e^{-\frac{\pi}{4}i}\right)^9$

$= \left(\sqrt{2}\right)^9 e^{-\frac{9\pi}{4}i}$

The principal argument $-\pi < \theta < \pi$

$$-\frac{9\pi}{4} \rightarrow -\frac{9\pi}{4} + 2\pi = -\frac{\pi}{4}$$

$$= 16\sqrt{2}e^{-\frac{\pi}{4}i}$$

$$= 16(1 - i)$$

QUESTION 2

(a) (i)

$$\int x^5 \cos(x^3)\ dx$$

Let $t = x^3$

$$\frac{dt}{dx} = 3x^2$$

$$\frac{dt}{3x^2} = dx$$

$$\int x^5 \cos(x^3)\ dx = \int x^5 \cos t \left(\frac{1}{3x^2}\right)\ dt$$

$$= \frac{1}{3}\int t \cos t\ dt$$

Using integration by parts

$u = t \quad \rightarrow \quad du = 1$

$dv = \cos t \quad \rightarrow \quad v = \sin t$

$$\frac{1}{3}\int t \cos t\ dt = \frac{1}{3}\left[t \sin t - \int \sin t\right]$$

$$= \frac{1}{3}[t \sin t + \cos t]$$

$$\therefore \int x^5 \cos(x^3)\ dx = \frac{1}{3}[x^3 \sin(x^3) + \cos(x^3)]$$

(a) (ii)

$$\int \frac{e^{2x}}{\sqrt{1 - e^{4x}}}\ dx$$

$$= \int \frac{e^{2x}}{\sqrt{1 - (e^{2x})^2}}\ dx$$

By recognition this corresponds to

$$\int \frac{u'}{\sqrt{1-u^2}}\,du = \sin^{-1} u \text{ where } u = e^{2x} \text{ and } u' = 2e^{2x}$$

$$\therefore \int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{\sqrt{1-(e^{2x})^2}}\,dx$$

$$= \frac{1}{2}\sin^{-1} e^{2x} + c$$

(b) (i)

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$x^4 + 1 = A(x^2+1)^2 + (Bx+C)(x^2+1)x + (Dx+E)x$$

$$= Ax^4 + 2Ax^2 + A + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$$

$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

Equating constants

$1 = A$

Equating coefficients of x^4

$1 = A + B$

$0 = B$

Equating coefficients of x^3

$0 = C$

Equating coefficients of x^2

$0 = 2A + B + D$

$-2 = D$

Equating coefficients of x

$0 = C + E$

$0 = E$

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{1}{x} - \frac{2x}{(x^2+1)^2}$$

(b) (ii)

$$\int \frac{x^4+1}{x(x^2+1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{2x}{(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x}\,dx - \int 2x(x^2+1)^{-2}\,dx$$

$$= \ln x - \frac{(x^2+1)^{-1}}{-1} + c$$

$$= \ln x + \frac{1}{x^2+1} + c$$

QUESTION 3

(a) (i)

$a_2 = \sqrt{2+\sqrt{2}}$

$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$

(a) (ii)

$P_n: a_n < a_{n+1}$ and $a_n \leq 3$

PART 1 – Proving that $a_n \leq 3$

$P_1: a_1 = \sqrt{2} < 3$

Therefore P_1 is true.

Assume P_n is true for $n = k$

$P_k: a_k \leq 3$

$P_{k+1}: a_{k+1} \leq 3$

Now

$a_{k+1} = \sqrt{2 + a_k}$

Assuming $a_k = 3$ then $a_{k+1} = \sqrt{2+3} = \sqrt{5}$

Since $\sqrt{5} < 3$ and $a_k \leq 3$, $a_{k+1} \leq 3$

Therefore P_{k+1} is true $\forall$ P_k is true.

PART 2 – Proving that $a_n < a_{n+1}$

$a_1 = \sqrt{2}$

$a_2 = \sqrt{2+\sqrt{2}}$

$a_1 < a_2$

Therefore, P_1 is true.

Assume P_n is true for $n = k$

$P_k : a_k < a_{k+1}$

$P_{k+1} : a_{k+1} < a_{k+2}$

Now

$a_{k+1} = \sqrt{2 + a_k}$

$a_{k+2} = \sqrt{2 + a_{k+1}}$

Since $a_k < a_{k+1}$, $a_{k+2} < a_{k+1}$

Therefore P_{k+1} is true whenever P_k is true

Hence by Mathematical Induction $a_n < a_{n+1}$ and $a_n \leq 3$ for all $n \in \mathbb{N}$.

(b) (i)

$f(x) = e^{-x^2} \quad \rightarrow \quad f(0) = e^{-0^2} = 1$

$f'(x) = -2xe^{-x^2} \quad \rightarrow \quad f'(0) = 0$

$f''(x) = -2e^{-x^2} + (-2x)\left(-2xe^{-x^2}\right) = -2e^{-x^2} + 4x^2e^{-x^2} \quad \rightarrow \quad f''(0) = -2$

$f'''(x) = 4xe^{-x^2} + 8xe^{-x^2} + 4x^2\left(-2xe^{-x^2}\right) = 12xe^{-x^2} - 8x^3e^{-x^2} \quad \rightarrow \quad f'''(0) = 0$

$f''''(x) = 12e^{-x^2} + 12x\left(-2xe^{-x^2}\right) - 24x^2e^{-x^2} - 8x^3\left(-2xe^{-x^2}\right) \quad \rightarrow \quad f''''(0) = 12$

$$f(x) \approx 1 - \frac{2}{2!}x^2 + \frac{12}{4!}x^4 \approx 1 - x^2 + \frac{1}{2}x^4$$

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} = \frac{(-1)^0 x^0}{0!} + \frac{(-1)^1 x^2}{1!} + \frac{(-1)^2 x^4}{2!} + \cdots$$

$$= 1 - x^2 + \frac{1}{2}x^4$$

(b) (ii)

Let $u_k = \frac{(-1)^k x^{2k}}{k!}$, then $u_{k+1} = \frac{(-1)^{k+1}\left(x^{2k+2}\right)}{(k+1)!}$

Series is valid

$$\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| < 1$$

$$\frac{u_{k+1}}{u_k} = \frac{(-1)^{k+1}(x^{2k+2})}{(k+1)!} \times \frac{k!}{(-1)^k x^{2k}} = \frac{(-1)x^2}{(k+1)}$$

$$\lim_{k\to\infty}\left|-\frac{x^2}{k+1}\right| < 1$$

$0 < 1$

Since this is true for all values of x, the series is valid for all values of x.

(c)

When $n = 1$: $\sin(1) - \sin\left(\frac{1}{2}\right)$

When $n = 2$: $\sin\left(\frac{1}{2}\right) - \sin\left(\frac{1}{3}\right)$

When $n = 3$: $\sin\left(\frac{1}{3}\right) - \sin\left(\frac{1}{4}\right)$

When $n = n - 1$: $\sin\left(\frac{1}{n-1}\right) - \sin\left(\frac{1}{n}\right)$

When $n = n$: $\sin\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n+1}\right)$

$$\sum\left(\sin\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n+1}\right)\right) = \sin(1) - \sin\left(\frac{1}{n+1}\right)$$

$$\lim_{n\to\infty}\left(\sin(1) - \sin\left(\frac{1}{n+1}\right)\right) = \sin(1) - \sin(0) = \sin 1$$

QUESTION 4

(a)

Coefficient of x^7 of $\left(x^2 - \frac{3}{x}\right)^8$

$$= \binom{8}{3}(x^2)^5\left(-\frac{3}{x}\right)^3$$

$$= 56x^{10}\left(-\frac{27}{x^3}\right)$$

$$= -1512x^7$$

Coefficient is -1512

(b)

$^{n}C_{r} + {}^{n}C_{r-1}$

$$= \frac{n!}{(n-r)!\,r!} + \frac{n!}{\big(n-(r-1)\big)!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,r(r-1)!} + \frac{n!}{(n-r+1)(n-r)!\,(r-1)!}$$

$$= \frac{n!\,(n-r+1)}{(n-r+1)(n-r)!\,r(r-1)!} + \frac{n!\,r}{(n-r+1)(n-r)!\,r(r-1)!}$$

$$= \frac{n!\,(n-r+1) + n!\,r}{(n-r+1)(n-r)!\,r(r-1)!}$$

$$= \frac{n!\,(n-r+1+r)}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)!}{(n+1-r)!\,r!}$$

$$= {}^{n+1}C_{r}$$

(c)(i)

Let $f(x) = 4\cos x - x^3 + 2$

$f(1) = 3.16121$

$f(1.5) = -1.09205$

$f(x)$ is continuous on the interval $[1, 1.5]$.

$$f(1) \times f(1.5) < 0$$

By the Intermediate Value Theorem there must be some $c \in [1, 1.5]$ such that $f(c) = 0$. Therefore, there is a root between 1 and 1.5.

(c) (ii)

$$\frac{1.5-\alpha}{\alpha-1} = \frac{1.09205}{3.16121}$$

$4.74182 - 3.16121\alpha = 1.09205\alpha - 1.09205$

$5.83387 = 4.25326\alpha$

$1.3716 = \alpha$

$f(1.3716) = 4\cos(1.3716) - (1.3716)^3 + 2 = 0.21115$

Root is now between 1.3716 and 1.5

$$\frac{1.5 - \alpha}{\alpha - 1.3716} = \frac{1.09205}{0.21115}$$

$0.21115(1.5 - \alpha) = 1.09205(\alpha - 1.3716)$

$0.316725 - 0.21115\alpha = 1.09205\alpha - 1.49786$

$1.814585 = 1.3032\alpha$

$1.3924 = \alpha$

$f(1.3924) = 4\cos(1.3924) - (1.3924)^3 + 2 = 0.0103$

Root is now between 1.3924 and 1.5

$$\frac{1.5 - \alpha}{\alpha - 1.3924} = \frac{1.09205}{0.0103}$$

$0.0103(1.5 - \alpha) = 1.09205(\alpha - 1.3924)$

$0.01545 - 0.0103\alpha = 1.09205\alpha - 1.5206$

$1.53605 = 1.1024\alpha$

$1.3934 = \alpha$

Since the last two approximations, correct to 2 decimal places, are both 1.39. The root is approximately 1.39.

(d) Let $f(x) = 3e^x + 2\ln x - 1$

$$f'(x) = 3e^x + \frac{2}{x}$$

$x_1 = 0.2$

$$x_2 = 0.2 - \frac{3e^{0.2} + 2\ln(0.2) - 1}{3e^{0.2} + \frac{2}{0.2}} = 0.2406$$

QUESTION 5

(a) (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.68 = 0.4 + 0.45 - P(A \cap B)$

$P(A \cap B) = 0.85 - 0.68 = 0.17$

If A and B are independent events $P(A) \times P(B) = P(A \cap B)$

(ii) $P(A) \times P(B) = 0.4 \times 0.45 = 0.18$

$P(A \cap B) = 0.17$

Therefore A and B are NOT independent.

(b) CASE 1: Mr Smith is on the committee, his wife is on the committee and 2 persons from the remaining 6

$$^6C_2 = 15$$

CASE 2: Mrs Smith is on the committee but not Mr Smith and 3 persons from the remaining 6

$$^6C_3 = 20$$

CASE 3: Neither Mr nor Mrs Smith is on the committee and 4 persons chosen from the remaining 6

$$^6C_4 = 15$$

Number of committees is $15 + 20 + 15 = 50$

(c) The number has to begin with 5, 6 or 7 and end with 3, 5 or 7

CASE 1: The number begins with 5 or 7

2 options for the first digit

4 options for the second digit

3 options for the third digit

2 options for the fourth digit

1 option for the fifth digit

2 options for the last digit!

$$2 \times 4! \times 2 = 96$$

CASE 2: The number begins with 6

1 option for the first digit

4 options for the second digit

3 options for the third digit

2 options for the fourth digit

1 option for the fifth digit

3 options for the last digit!

$$4! \times 3 = 72$$

Total number of numbers is $96 + 72 = 168$

(d) (i) $AB = \begin{pmatrix} 5 & -2 & 3 \\ 0 & 3 & -4 \\ 2 & 0 & 6 \end{pmatrix}\begin{pmatrix} 18 & 12 & -1 \\ -8 & 24 & 20 \\ -6 & -4 & 15 \end{pmatrix} = \begin{pmatrix} 88 & 0 & 0 \\ 0 & 88 & 0 \\ 0 & 0 & 88 \end{pmatrix}$

$AB = 88I$

$A\left[\frac{1}{88}B\right] = I$

$A^{-1} = \frac{1}{88}B$

(d) (ii)

$$\begin{pmatrix} 5 & -2 & 3 \\ 0 & 3 & -4 \\ 2 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{88}\begin{pmatrix} 18 & 12 & -1 \\ -8 & 24 & 20 \\ -6 & -4 & 15 \end{pmatrix}\begin{pmatrix} 7 \\ 11 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{88}\begin{pmatrix} 264 \\ 88 \\ -176 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

QUESTION 6

(a)

(i) $y' \cos x = y \sin x + \sin 2x$

$y' \cos x - y \sin x = \sin 2x$

This is an EXACT differential equation

$$\int y' \cos x - y \sin x = \int \sin 2x \ dx$$

$$y \cos x = -\frac{1}{2}\cos 2x + c$$

$$y \cos x = -\frac{2\cos^2 x - 1}{2} + C$$

$$y \cos x = \frac{1}{2} - \cos^2 x + C$$

$$y = \frac{1}{2\cos x} - \cos x + \frac{C}{\cos x}$$

$y = \frac{1}{2}\sec x - \cos x + C\sec x$

(ii) $y(0) = 0$

$0 = \frac{1}{2}\sec(0) - \cos(0) + C\sec(0)$

$0 = \frac{1}{2}(1) - 1 + C$

$C = \frac{1}{2}$

$y = \frac{1}{2}\sec x - \cos x + \frac{1}{2}\sec x$

$y = \sec x - \cos x$

(b) (i) $y'' + 2y' + y = xe^{-x}$

Auxiliary equation

$u^2 + 2u + 1 = 0$

$(u+1)^2 = 0$

$u = -1, -1$

$y = e^{-x}(Cx + D)$

(ii) Let $y = (Ax^3 + Bx^2)e^{-x}$

$y' = (3Ax^2 + 2Bx)e^{-x} - (Ax^3 + Bx^2)e^{-x}$

$y'' = (6Ax + 2B)e^{-x} - (3Ax^2 + 2Bx)e^{-x} - (3Ax^2 + 2Bx)e^{-x} + (Ax^3 + Bx^2)e^{-x}$

Substituting these expressions into $y'' + 2y' + y = xe^{-x}$

$(6Ax + 2B)e^{-x} - (3Ax^2 + 2Bx)e^{-x} - (3Ax^2 + 2Bx)e^{-x} + (Ax^3 + Bx^2)e^{-x}$

$+2\left((3Ax^2 + 2Bx)e^{-x} - (Ax^3 + Bx^2)e^{-x}\right) + (Ax^3 + Bx^2)e^{-x} = xe^{-x}$

Equating e^{-x} terms

$2Be^{-x} = 0e^{-x}$

$\rightarrow B = 0$

Equating xe^{-x} terms

$(6A - 2B - 2B + 4B)xe^{-x} = xe^{-x}$

$\rightarrow\ 6A = 1$

$$A = \frac{1}{6}$$

Particular Integral

$$y = e^{-x}(Cx + D) + \frac{1}{6}x^3e^{-x}$$