CAPE 2018 SOLUTIONS

QUESTION 1

(a) (i)

p	q	$\sim p$	$\sim q$	$(p \vee q)$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	**F**	**F**	**T**	**F**	**F**
T	F	**F**	**T**	**T**	**F**	**F**
F	T	**T**	**F**	**T**	**F**	**F**
F	F	**T**	**T**	**F**	**T**	**T**

(ii) $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent because they have the same truth table values.

(b) (i) $5 \oplus 2 = 2(5) + 3(2) = 16$

(ii) $2a \in \mathbb{R}, 3b \in \mathbb{R}$ and the sum of two real numbers is a real number. Therefore, $\oplus$ is closed on $\mathbb{R}$.

(iii) If $\oplus$ is commutative $a \oplus b = b \oplus a$.

$b \oplus a = 2b + 3a$

Therefore, $a \oplus b \neq b \oplus a$ so $\oplus$ is NOT commutative.

(c) $(2x + a)(x - 1)(bx + 1)$

$= (2x + a)(bx^2 - bx + x - 1)$

$= 2bx^3 - 2bx^2 + 2x^2 - 2x + abx^2 - abx + ax - a$

$= 2bx^3 + (-2b + 2 + ab)x^2 + (-2 - ab + a)x - a$

$= cx^3 + 10x^2 - 2x - 10$

Equating constants

$a = 10$

Equating coefficients of x

$-2 - ab + a = -2$

$-2 - 10b + 10 = -2$

$b = 1$

Equating coefficients of x^3

$2b = c$

$2 = c$

(d) $\log_4(2x + 2) - \log_2(x + 1) = 1$

$$\frac{\log_2(2x + 2)}{\log_2 4} - \log_2(x + 1) = \log_2 2$$

$$\frac{\log_2(2x + 2)}{2} - \log_2(x + 1) = \log_2 2$$

$$\log_2(2x + 2) - 2\log_2(x + 1) = 2\log_2 2$$

$$\log_2(2x + 2) - \log_2(x + 1)^2 = \log_2 4$$

$$\frac{2x + 2}{(x + 1)^2} = 4$$

$$2x + 2 = 4(x + 1)^2$$

$$2x + 2 = 4(x^2 + 2x + 1)$$

$$2x + 2 = 4x^2 + 8x + 4$$

$$0 = 4x^2 + 6x + 2$$

$$2x^2 + 3x + 1 = 0$$

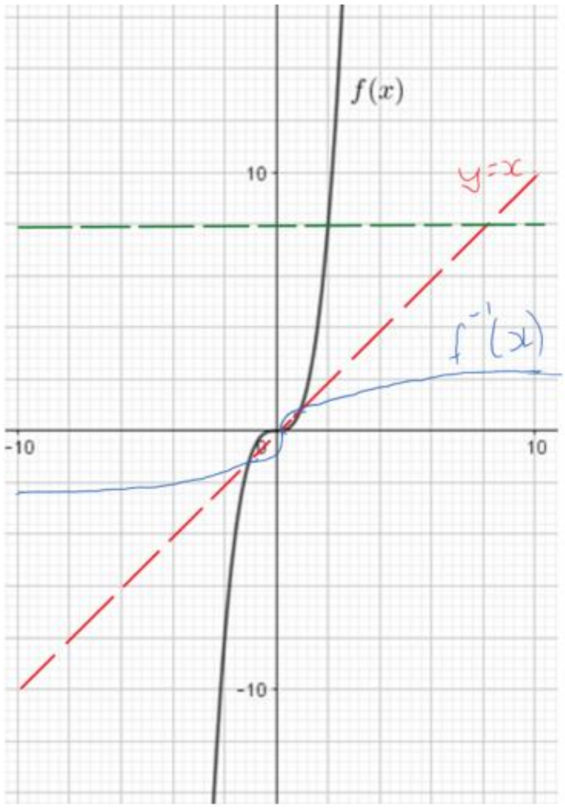
$$(2x + 1)(x + 1) = 0$$

$$x = -1, -\frac{1}{2}$$

$x = -1$ is INVALID therefore $x = -\frac{1}{2}$

QUESTION 2

(a)



f passes the Horizontal Line test and it is therefore injective. f^{-1} exists and this implies that f is surjective. Consequently, f is bijective.

(b) $|x-y| \leq |x-z| + |z-y|$

(i) $x - y = x - z + z - y$

$x - y = (x - z) + (z - y)$

Taking modulus of both sides

$|x - y| = |(x - z) + (z - y)|$

If $x - z \leq 0$

$|x - z| \geq x - z$

If $z - y \leq 0$

$|z - y| \geq z - y$

$|x-y| \leq \big||x-z|+|z-y|\big|$

$|x-y| \leq |x-z|+|z-y|$

(ii) $|6x-2|+x^2 \leq 5$

$|6x-2| \leq 5-x^2$

$-(5-x^2) \leq 6x-2 \leq 5-x^2$

$-5+x^2-6x+2 \leq 0 \leq 5-x^2-6x+2$

$x^2-6x-3 \leq 0 \leq -x^2-6x+7$

We have two inequalities to evaluate

CASE 1: $x^2-6x-3 \leq 0$

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-3)}}{2(1)}$$

$x=3+2\sqrt{3}=6.46$

$x=3-2\sqrt{3}=-0.464$

$-0.464 \leq x \leq 6.46$

CASE 2: $0 \leq -x^2-6x+7$

$x^2+6x-7 \leq 0$

$x^2+6x-7 \leq 0$

$(x+7)(x-1) \leq 0$

Roots are $x=-7, 1$

$-7 \leq x \leq 1$

Combining the inequalities we get

$-0.464 \leq x \leq 1$

(c) $2x^3 - x^2 + 1 = 0$

$\alpha + \beta + \gamma = \frac{1}{2}$

$\alpha\beta + \beta\gamma + \alpha\gamma = 0$

$\alpha\beta\gamma = -\frac{1}{2}$

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

$= \frac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2}{\alpha^2\beta^2\gamma^2}$

$= \frac{(\alpha\gamma + \alpha\beta + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)}{(\alpha\beta\gamma)^2}$

$= \frac{(0)^2 - 2\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)^2}$

$= 2$

$\left(\frac{1}{\alpha^2}\right)\left(\frac{1}{\beta^2}\right) + \left(\frac{1}{\alpha^2}\right)\left(\frac{1}{\gamma^2}\right) + \left(\frac{1}{\beta^2}\right)\left(\frac{1}{\gamma^2}\right)$

$= \frac{\alpha^2 + \beta^2 + \gamma^2}{(\alpha\beta\gamma)^2}$

$= \frac{(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)}{(\alpha\beta\gamma)^2}$

$= \frac{\left(\frac{1}{2}\right)^2 - 2(0)}{\left(-\frac{1}{2}\right)^2}$

$= 1$

$\left(\frac{1}{\alpha^2}\right)\left(\frac{1}{\beta^2}\right)\left(\frac{1}{\gamma^2}\right)$

$= \frac{1}{(\alpha\beta\gamma)^2}$

$$= \frac{1}{\left(-\frac{1}{2}\right)^2}$$

$$= 4$$

Equation is $x^3 - 2x^2 + x - 4 = 0$

QUESTION 3

(a) (i)

$L.H.S$

$$\frac{\sin 2\theta - \cos 2\theta + 1}{\cos 2\theta + \sin 2\theta - 1}$$

$$= \frac{2\sin\theta\cos\theta - (1 - 2\sin^2\theta) + 1}{(2\cos^2\theta - 1) + 2\sin\theta\cos\theta - 1}$$

$$= \frac{2\sin\theta\cos\theta + 2\sin^2\theta}{2\cos^2\theta + 2\sin\theta\cos\theta - 2}$$

$$= \frac{2\sin\theta\,(\cos\theta + \sin\theta)}{2\cos^2\theta + 2\sin\theta\cos\theta - 2(\sin^2\theta + \cos^2\theta)}$$

$$= \frac{2\sin\theta\,(\cos\theta + \sin\theta)}{2\cos^2\theta + 2\sin\theta\cos\theta - 2\sin^2\theta - 2\cos^2\theta}$$

$$= \frac{2\sin\theta\,(\cos\theta + \sin\theta)}{2\sin\theta\,(\cos\theta - \sin\theta)}$$

$$= \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$$

$$= \frac{(\cos\theta + \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$$

$$= \frac{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta}{(\cos^2\theta - \sin^2\theta)}$$

$$= \frac{1 + \sin 2\theta}{\cos 2\theta}$$

$$= \frac{1}{\cos 2\theta} + \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \sec 2\theta + \tan 2\theta$$

$$= R.H.S$$

(ii) Replacing 2θ with θ in part (i) we get

$$\frac{\sin\theta - \cos\theta + 1}{\cos\theta + \sin\theta - 1} = \sec\theta + \tan\theta$$

$$\therefore \sec\theta + \tan\theta = 0$$

$$\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = 0$$

$$\frac{1 + \sin\theta}{\cos\theta} = 0$$

$$1 + \sin\theta = 0$$

$$\sin\theta = -1$$

Reference angle is $\sin^{-1}(1) = \frac{\pi}{2}$

Sine is negative in III and IV

III: $\theta = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$

IV: $\theta = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$

General solution: $\theta = \frac{3\pi}{2} + 2n\pi, \;\; n \in \mathbb{Z}$

(b) $\cos A = \frac{3}{5}$ and $\sin B = \frac{3}{4}$

By Pythagoras' Theorem

$$\sin A = \frac{4}{5} \quad \text{and} \quad \cos B = \frac{\sqrt{7}}{4}$$

(i) $\sin 2A = 2\sin A\cos A$

$$\sin 2A = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$$

(ii) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

$$\cos(A + B) = \left(\frac{3}{5}\right)\left(\frac{\sqrt{7}}{4}\right) - \left(\frac{4}{5}\right)\left(\frac{3}{4}\right)$$

$$\cos(A + B) = \frac{3\sqrt{7} - 12}{20}$$

(c) $\sin\theta - \sqrt{3}\cos\theta = 1$

Re – writing in the form $r\sin(\theta - \alpha)$

$$r = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = 2$$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$2\sin\left(\theta - \frac{\pi}{3}\right) = 1$$

$$\sin\left(\theta - \frac{\pi}{3}\right) = \frac{1}{2}$$

Reference angle is $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Sine is positive in I and II

I: $\theta - \frac{\pi}{3} = \frac{\pi}{6} \quad \rightarrow \quad \theta = \frac{\pi}{2}$

II: $\theta - \frac{\pi}{3} = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \quad \rightarrow \quad \theta = \frac{7\pi}{6}$ OUTSIDE OF RANGE

To determine the corresponding value for $\frac{7\pi}{6}$ within the given range we subtract 2π.

$$\theta = \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$$

Therefore $\theta = -\frac{5\pi}{6}, \frac{\pi}{2}$

QUESTION 4

(a) (i) $x^2 + y^2 - 2x + 2y + 1 = 0$

$x^2 - 2x + 1 + y^2 + 2y + 1 = -1 + 1 + 1$

$(x-1)^2 + (y+1)^2 = 1$

Centre is $(1, -1)$ and radius is 1.

(ii) If $(1, -2)$ is a point of intersection it must satisfy both equations

$y = x - 3$

$-2 = 1 - 3$

$-2 = -2$

TRUE

$x^2 + y^2 - 2x + 2y + 1 = 0$

$1^2 + (-2)^2 - 2(1) + 2(-2) + 1 = 0$

$1 + 4 - 2 - 4 + 1 = 0$

$0 = 0$

TRUE

Therefore, $(1, -2)$ is a point of intersection.

(iii) Gradient of radius using $(1, -1)$ and $(1, -2)$

$$m = \frac{-1 - (-2)}{1 - 2} = \text{undefined}$$

Therefore, gradient of tangent is 0.

Equation of tangent is $y = -2$.

(b) $u = \begin{pmatrix} s \\ 3 \\ s \end{pmatrix}$ and $v = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix}$

If u and v are parallel $v = ku$ where k is a constant.

$$k = \frac{6}{3} = 2$$

$v = 2u$

Therefore $-2 = 2s \quad \rightarrow \quad -1 = s$

$4 = 2s \quad \rightarrow \quad 2 = s$

Since s cannot have to different values u and v are NOT parallel.

(c) $r.n = a.n$

$$r.\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}.\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

$$r.\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} = (1)(2) + (3)(4) + (0)(5)$$

$$r.\begin{pmatrix}2\\4\\5\end{pmatrix} = 14$$

QUESTION 5

(a) $u = x^4 + 2 \quad \rightarrow \quad x^4 = u - 2$

$$\frac{du}{dx} = 4x^3$$

$$dx = \frac{du}{4x^3}$$

$$\int (x^4 + 2)^3 (4x^7)\ dx$$

$$= \int u^3 (4x^7) \left(\frac{du}{4x^3}\right)$$

$$= \int u^3 (x^4)\ du$$

$$= \int u^3 (u - 2)\ du$$

$$= \int u^4 - 2u^3\ du$$

$$= \frac{u^5}{5} - \frac{2u^4}{4} + c$$

$$= \frac{(x^4 + 2)^5}{5} - \frac{(x^4 + 2)^4}{2} + c$$

(b) Solving simultaneously for x

$y = x^2 \qquad (1)$

$\frac{1}{8}y^2 = x \qquad (2)$

Sub (1) into (2)

$$\frac{1}{8}(x^2)^2 = x$$

$$x^4 = 8x$$

$$x^4 - 8x$$

$x(x^3 - 8) = 0$

$x = 0$

$x^3 - 8 = 0$

$x^3 = 8$

$x = 2$

From (2): $y = (8x)^{\frac{1}{2}}$

$AREA = \int_0^2 (8x)^{\frac{1}{2}} - x^2 \; dx$

$$= \left[\frac{(8x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(8)} - \frac{x^3}{3} \right]_0^2$$

$$= \left[\frac{(8x)^{\frac{3}{2}}}{12} - \frac{x^3}{3} \right]_0^2$$

$$= \frac{\left(8(2)\right)^{\frac{3}{2}}}{12} - \frac{2^3}{3}$$

$$= \frac{8}{3} \text{ units}^2$$

(c) $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x$

(i) $f'(x) = 12x^3 - 6x^2 - 12x + 6$

(ii) $f''(x) = 36x^2 - 12x - 12$

(iii) $f'(x) = 0$ for stationary points

$12x^3 - 6x^2 - 12x + 6 = 0$

$2x^3 - x^2 - 2x + 1 = 0$

By inspection $x = 1$ is a root and therefore $x - 1$ is a factor

	x^3	x^2	x	constant
	2	-1	-2	1
		-2	-1	1
$x-1$	2	1	-1	0

$2x^3 - x^2 - 2x + 1 = 0$

$(x-1)(2x^2 + x - 1) = 0$

$(x-1)(2x-1)(x+1) = 0$

$x = -1, \frac{1}{2}, 1$

$f''(-1) = 36(-1)^2 - 12(-1) - 12 = 36 \quad \rightarrow \quad$ minimum

$f''\left(\frac{1}{2}\right) = 36\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right) - 12 = -9 \quad \rightarrow \quad$ maximum

$f''(1) = 36(1)^2 - 12(1) - 12 = 12 \quad \rightarrow \quad$ minimum

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QUESTION 6

(a) $f(x) = \begin{cases} \frac{x^4-1}{x-1} & x < 1 \\ 4x & x > 1 \\ 2 & x = 1 \end{cases}$

(i) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^4-1}{x-1} = \lim_{x \to 1^-} x^3 + x^2 + x + 1 = 1^3 + 1^2 + 1 + 1 = 4$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x = 4(1) = 4$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$ the limit of f at $x = 1$ exists

(ii) $f(1) = 2$

Since $\lim_{x \to 1} f(x) \neq f(1)$, f is not continuous at $x = 1$.

(b) $x = 2\cos\theta, y = 3 - \sin\theta$

(i) $\frac{dx}{d\theta} = -2\sin\theta$

$\frac{dy}{d\theta} = -\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{dt}{d\theta}$$

$$\frac{dy}{dx} = \frac{-\cos\theta}{-2\sin\theta} = \frac{1}{2}\cot\theta$$

(ii) When $x = \sqrt{3}$

$$\sqrt{3} = 2\cos\theta$$

$$\frac{\sqrt{3}}{2} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

$$\frac{dy}{dx} = \frac{1}{2}\cot\left(\frac{\pi}{6}\right) = \frac{1}{2\tan\left(\frac{\pi}{6}\right)} = \frac{\sqrt{3}}{2}$$

Gradient of normal is $-\frac{2}{\sqrt{3}}$.

$$y = mx + c \quad m = -\frac{2}{\sqrt{3}} \quad \left(\sqrt{3}, \frac{5}{2}\right)$$

$$\frac{5}{2} = -\frac{2}{\sqrt{3}}\left(\sqrt{3}\right) + c$$

$$\frac{9}{2} = c$$

$$y = -\frac{2}{\sqrt{3}}x + \frac{9}{2}$$

(c) (i) $\frac{dy}{dx} = x\left(\frac{1}{y}\right)$

$$y\ dy = x\ dx$$

$$\int y\ dy = \int x\ dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$y^2 = x^2 + C$$

(ii) $y^2 = x^2 + C$

$$3^2 = 1^2 + C$$

$$8 = C$$

$$y^2 = x^2 + 8$$