CAPE Pure Mathematics 2019 Unit 1

1 $f(x) = ax^2 + 12x + b$

$f(3) = 9a + 36 + b = 0$ ①

$f(-6) = 36a - 72 + b = -27$ ②

Solving simultaneously ② − ①

$27a - 108 = -27$

$27a = 81$

$a = 3$

Substituting $a=3$ into ①

$9(3) + 36 + b = 0$

$b = -63$

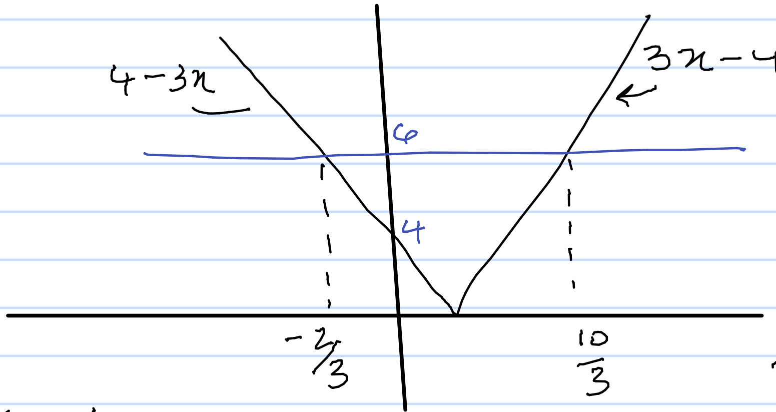
(ii) $f(x) = 3x^2 + 12x - 63$

$= 3(x^2 + 4x - 21)$

$= 3(x+7)(x-3)$

So factors of $f(x)$ are 3, $(x-3)$ and $(x+7)$

(b) $|3x-4| \le 6$



$3x-4=6$

$3x=10$

$x=10/3$

$4-3x=6$

$-3x=2$

$x=-2/3$

so solution $-\frac{2}{3} \le x \le \frac{10}{3}$

(c) $a * b = \frac{ab}{2}$

now $ab = ba$ multiplication is commutative

$\Rightarrow a*b = b*a \Rightarrow *$ is commutative

(d) Let P_n be the proposition that

$5^n - 1$ is divisible by 4 $\forall n \in \mathbb{N}$

i.e. $5^n - 1 = 4m \quad m \in \mathbb{Z}^+$

for $n=1$: $5^1 - 1 = 4$ which is divisible by 4

so P_1 is true

Assume that P_n is true for $n=k$, $k \geq 1$

i.e $5^k - 1 = 4m \Rightarrow 5^k = 4m+1$

let $n = k+1$

P_{k+1}: $5^{k+1} - 1 = 5.5^k - 1$

$= 5(4m+1) - 1$

$= 20m + 4 = 4(5m+1)$

So P_{k+1} is true when P_k is true and

since P_1 is true by mathematical

induction P_n is true $\forall n \in \mathbb{N}$.

2. (a) $A = \{x : x \in \mathbb{R}, x \geq 1\}$

Given $f : A \to R$, $f(x) = x^2 - x$

f is 1−1 if $f(x_1) = f(x_2) \Rightarrow x = y$

$f(x_1) = x_1^2 - x_1$

$f(x_2) = x_2^2 - x_2$

$f(x_1) = f(x_2)$: $x_1^2 - x_1 = x_2^2 - x_2$

$x_1^2 - x_2^2 = x_1 - x_2$

$(x_1 - x_2)(x_1 + x_2) = (x_1 - x_2)$

$(x_1 - x_2)(x_1 + x_2) - (x_1 - x_2) = 0$

$(x_1 - x_2)(x_1 + x_2 - 1) = 0$

so either $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

or $x_1 + x_2 - 1 = 0 \Rightarrow x_1 = 1 - x_2$

Since $x_1, x_2 \in A$ i.e. $x_1, x_2 \geq 1$

$x_1 = 1 - x_2$ not possible

So $x_1 = x_2$

$\therefore$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\therefore$ f is 1−1

2(b) $f(x) = 3x+1$

$x = 3y+1$

$y = \frac{x-1}{3} \Rightarrow f^{-1}(x) = \frac{x+1}{3}$

(ii) $f^{-1} \circ g = \frac{e^{3x}+1}{3}$

2(c) $3 - \frac{4}{9^x} - \frac{4}{81^x} = 0$

$3 - \frac{4}{9x} - \frac{4}{9^{2x}} = 0$

$3(9^{2x}) - 4(9^x) - 4 = 0$

let $u = 9^x$

$3u^2 - 4u - 4 = 0$

$(3u+2)(u-2) = 0$

$u = -2/3$ or $u = 2$

$9^x = -2/3$ no solution

$9^x = 2 \Rightarrow x = \frac{\ln 2}{\ln 9}$

2(d) $2x^3 - 5x^2 + 4x + 6 = 0$

$$\alpha + \beta + \gamma = \frac{5}{2}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{4}{2} = 2$$

$$\alpha\beta\gamma = -\frac{6}{2} = -3$$

(ii) $\alpha^2 + \beta^2 + \gamma^2 = \left(\frac{5}{2}\right)^2 - 2(2) = \frac{25}{4} - 4 = \frac{9}{4}$

$$(\alpha\beta)^2 + (\alpha\gamma)^2 + (\beta\gamma)^2 = 2^2 - 2 \cdot -3 \cdot \frac{5}{2}$$

$$= 4 + 15 = 19$$

$$(\alpha\beta\gamma)^2 = (-3)^2 = 9$$

equation with roots α^2, β^2 and γ^2

$$x^2 - \frac{9}{4}x^2 + 19x - 9 = 0$$

or $4x^2 - 9x^2 + 76x - 36 = 0$

3 (a) $5\sec\theta - 2\sec^2\theta = \tan^2\theta - 1$

$2\sec^2\theta - 5\sec\theta + \tan^2\theta - 1 = 0$

$2\sec^2\theta - 5\sec\theta + \sec^2\theta - 1 - 1 = 0$

$3\sec^2\theta - 5\sec\theta - 2 = 0$

$(3\sec\theta + 1)(\sec\theta - 2) = 0$

$\sec\theta = -\frac{1}{3}$ no solution

$\sec\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$ $\theta = -\frac{\pi}{3}$

(b)(i) $f(x) = \cos x + 2\sin x$

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\cos\alpha = 2$
$R\sin\alpha = 1$

$R = \sqrt{2^2+1} = \sqrt{5}$

$\tan\alpha = \frac{1}{2} \Rightarrow \alpha = 0.464^c$

so $f(x) = \sqrt{5}\sin(x + 0.464)$

(ii) $\sqrt{5}\sin(x+0.464)$ 0

$\sin(x+0.464) = 0$

$\alpha = 0$

$x + 0.464 = n\pi$

$x = n\pi - 0.464$

3(b)(iii) minimum value of $\frac{2}{2-f}$

occurs when $2-f$ is at a maximum

i.e. when $f = -\sqrt{5}$

minimum value $= \frac{2}{2-(-\sqrt{5})} = \frac{2}{2+\sqrt{5}}$

3(c) $\tan(A+B) - \tan A$

$$= \frac{\sin(A+B)}{\cos(A+B)} - \frac{\sin A}{\cos A}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos(A+B)} - \frac{\sin A}{\cos A}$$

$$= \frac{\cos A \sin A \cos B + \cos^2 A \sin B - \sin A\,(\cos(A+B))}{\cos A \cos(A+B)}$$

$$= \frac{\cos A \sin A \cos B + \cos^2 A \sin B - \sin A \cos A \cos B + \sin^2 A \sin B}{\cos A \cos(A+B)}$$

$$= \frac{\cos^2 A \sin B + \sin^2 A \sin B}{\cos A \cos(A+B)}$$

$$= \frac{\sin B\,(\cos^2 A + \sin^2 A)}{\cos A \cos(A+B)}$$

$$= \frac{\sin B}{\cos A \cos(A+B)} = \text{R.H.S.}$$

4 (a) $x^2 + y^2 - 2x - 4y - 5 = 0$

$x^2 - 2x + y^2 - 4y - 5 = 0$

$x^2 - 2x + (-1)^2 + y^2 - 4y + (-2)^2 - 5 - (-1)^2 - (-2)^2 = 0$

$(x-1)^2 + (y-2)^2 - 10 = 0$

coordinates of C : $(1, 2)$

(b) substitute $y = 2x + 5$

$x^2 + (2x+5)^2 - 2x - 4(2x+5) - 5 = 0$

$x^2 + 4x^2 + 20x + 25 - 2x - 8x - 20 - 5 = 0$

$5x^2 + 10x = 0$

$x^2 + 2x = 0$

$x(x+2) = 0 \Rightarrow x = 0 \quad x = 2$

at $x = 0$ $y = 5$ $(0, 5)$ A

at $x = -2$ $y = 1$ $(-2, 1)$ B

using $C(1, 2)$

$m_{BC} = \dfrac{1-2}{-2-1} = \dfrac{-1}{-3} = \dfrac{1}{3}$

$m_{AC} = \dfrac{5-2}{0-1} = \dfrac{3}{-1} = -3$

$m_{BC} \times m_{AC} = \dfrac{1}{3} \times -3 = -1$

$\Rightarrow$ BC and AC are perpendicular.

(iii) $m_{AC} = -3$

gradient of tangent at A $= \frac{1}{3}$

so equation of tangent at A

$(y-5) = \frac{1}{3}(x-0)$

$y = \frac{1}{3}x + 5$

OR

$m_{BC} = \frac{1}{3}$

gradient of tangent at B $= -3$

equation of tangent at B$(-2, 1)$

$(y-1) = -3(x-(-2))$

$y-1 = -3x-6$

$y = -3x - 5$

4.(b) (i)

$\overrightarrow{PQ} = -2\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k}$

$\overrightarrow{QR} = -2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

(ii) If $\underset{\sim}{s} = -16j - 8k$ is $\perp$ to the plane with P Q & R.

the $s \cdot \overrightarrow{PQ} = 0$ or $s \cdot \overrightarrow{QR} = 0$

$(-16j - 8k) \cdot (-2i + 3j - 6k)$ or $(-16j - 8k) \cdot (-2i - j + 2k)$

$= -48 + 48 = 0$ $\qquad = 16 - 16 = 0$

(iii) cartesian equation of plane

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad a = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \quad n = \begin{pmatrix} 0 \\ -16 \\ -8 \end{pmatrix}$$

$$(r - a) \cdot n = 0$$

$$\begin{pmatrix} x - 3 \\ y + 1 \\ z - 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -16 \\ -8 \end{pmatrix} = 0$$

$$-16y - 16 - 8z + 16 = 0$$

$$-16y - 8z = 0$$

$$-2y - z = 0$$

or $\quad 2y - z = 0$

5. (a) (i) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{-x} = 0$

(ii) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\Rightarrow \lim_{x\to 0} f(x)$ does NOT exist

(iii) $f(1) = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x} = 1$

Since $\lim_{x\to 1} f(x) = 1$ and $\lim_{x\to 1} f(x) = f(1)$

$f(x)$ is continuous at $x = 1$

(b) $y = x\sqrt{x} = x^{3/2}$

$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$

the line $3x - y + 6 = 0 \Rightarrow y = 3x + 6$

so gradient $= 3$

$\frac{3}{2} x^{1/2} = 3 \qquad x^{1/2} = 2 \Rightarrow x = 4$

at $x = 4$ $\quad y = 4\sqrt{4} = 8$ so the point is $(4, 8)$

5(c) $y = \sin^2(\cos x) =$

$$\frac{dy}{dx} = 2\sin(\cos x)\cos(\cos x)(-\sin x)$$

$$= \sin(2\cos x)(-\sin x)$$

So $\sin x = 0$

$\alpha = 0$

$x = n\pi$

or $\sin(2\cos x) = 0$

$2\cos x = 0$

$\alpha = \frac{\pi}{2}$

$x = 2n\pi \pm \frac{\pi}{2}$

(ii) $$\frac{dy}{dx} = -\sin x \sin(2\cos x)$$

$$\frac{d^2y}{dx} = -\sin x\left[\cos(2\cos x)(-2\sin x)\right]$$

$$+ \sin(2\cos x)(-\cos x)$$

at $x = n\pi$ (n odd)

$$\frac{d^2y}{dx} = 0 + \sin(-2) < 0 \Rightarrow \text{maximum}$$

at $x = n\pi$ (n even)

$$\frac{d^2y}{dx^2} = 0 + \sin(2)\cdot(-1) < 0 \Rightarrow \text{maximum}$$

(6) $\int x(1-x)^2\,dx$

$u = 1-x$

$\frac{du}{dx} = -1 \Rightarrow du = -dx$

$x = 1-u$

So on substituting integral becomes

$$\int (1-u)\,u^2\,(-du) = \int u^3 - u^2\,du$$

$$= \frac{u^4}{4} - \frac{u^3}{3} + C$$

$$= \frac{(1-x)^4}{4} - \frac{(1-x)^3}{3} + C$$

(b) $f(t) = 2\cos t \qquad g(t) = 4\sin 5t + 3\cos t$

$$\int [f(t) + g(t)]\,dt = \int 4\sin 5t + 5\cos t\;dt$$

$$= -\frac{4}{5}\cos 5t + 5\sin t + C$$

$$\int f(t)\,dt = \int 2\cos t\,dt = 2\sin t + C \quad (1)$$

$$\int g(t)\,dt = \int 4\sin 5t + 3\cos t\,dt$$

$$= -\frac{4}{5}\cos t + 3\sin t + C \quad (2)$$

$$(1) + (2) = -\frac{4}{5}\cos t + 5\sin t + C$$

(c) Volume $= \pi \int_{\frac{\pi}{2}}^{\pi} \sin^2 x \, dx$

$$= \frac{\pi}{2} \int_{\pi/2}^{\pi} 1 - \cos 2x \, dx$$

$$= \frac{\pi}{2} \left[x - \frac{\sin 2x}{2} \right]_{\pi/2}^{\pi}$$

$$= \frac{\pi}{2} \left[(\pi - 0) - \left(\frac{\pi}{2} - 0 \right) \right]$$

$$= \frac{\pi^2}{4}$$

6 (d) $\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{y}$

$y\,dy = x\sqrt{x^2+1}\,dx$

integrate both sides

$\int y\,dy = \int x\sqrt{x^2+1}\,dx$

let $u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{y^2}{2} + c = \int x\sqrt{x^2+1}\,dx$

$= \int \frac{u^{1/2}}{2}\,du = \frac{u^{3/2}}{3} + c$

$\frac{y^2}{2} = \frac{1}{3}(x^2+1)^{3/2} + C$

$y^2 = \frac{2}{3}(x^2+1)^{3/2} + C$

when $x=0$ $y=2$

$4 = \frac{2}{3} + C$

$C = \frac{10}{3}$

hence $y^2 = \frac{2}{3}(x^2+1)^{3/2} + \frac{10}{3}$