2016 Cape Unit 1 Paper 2 Solutions

1. (a) (i) $2(-3)^3-(-3)^2+p(-3)+q=0$

$-54-9-3p+q=0$

$-3p+q=63$ …eqn. 1

$2(-1)^3-(-1)^2+p(-1)+q=10$

$-2-1-p+q=10$

$-p+q=13$ …eqn. 2

eqn. 1 – eqn. 2 $\Rightarrow -2p=50$

$p=-25$

$q=13+p=13-25=-12$

(ii) $f(x)=2x^3-x^2-25x-12$

$$\begin{array}{r} 2x^2-7x-4 \\ x+3\,\overline{\big)\,2x^3-x^2-25x-12} \\ \underline{2x^3+6x^2} \\ -7x^2-25x \\ \underline{-7x^2-21x} \\ -4x-12 \\ \underline{-4x-12} \end{array}$$

$2x^2-7x-4\equiv(2x+1)(x-4)$

$\therefore\ f(x)=(x+3)(2x+1)(x-4)$

$(x+3)(2x+1)(x-4)=0$

$x=-3,\ 4,\ -\frac{1}{2}$

(b) Let $P_n$ be the proposition that $6^n-1$ is divisible by 5

$P_1: 6^1-1=5$, therefore true for $n=1$

$P_k: 6^k-1=5M,\quad M\in\mathbb{N}$

$P_{k+1}: 6^{k+1}-1 \quad =6(6^k)-1$

$=5(6^k)+(6^k-1)$

where both are divisible by 5

Since $P_k$ being true implies $P_{k+1}$ is also true, by mathematical induction $6^n-1$ is divisible by 5 $\forall n\in\mathbb{N}$.

(c) (i)

| $p$ | $q$ | $p\to q$ | $p\vee q$ | $p\wedge q$ | $(p\vee q)\to(p\wedge q)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | T | T | F | F |
| F | F | T | F | F | T |

(ii) They are not logically equivalent since the truth values are not identical.

2. (a) $\log_2(10-x)+\log_2 x = 4$

$\log_2 x(10-x) = 4$

$10x - x^2 = 2^4$

$x^2 - 10x + 16 = 0$

$(x-2)(x-8) = 0$

$x = 2, 8$

(b) for $f$ to be injective $f(a) = f(b) \Rightarrow a = b$

$$\frac{a+3}{a-1} = \frac{b+3}{b-1}$$

$(a+3)(b-1) = (b+3)(a-1)$

$ab - a + 3b - 3 = ab - b + 3a - 3$

$-a + 3b = -b + 3a$

$4b = 4a$

$b = a$

$f$ is onto if for every $y \in \mathbb{R}$ there exist $x \in \mathbb{R}$ such that $f(x) = y$

let $y = \dfrac{x+3}{x-1}$

$y(x-1) = x+3$

$xy - y = x + 3$

$x(y-1) = y+3$

then $x = \dfrac{y+3}{y-1}$

since y =1 is NOT in the codomain as f is a self inverse function then f is ONTO

(c) (i) $\alpha + \beta + \gamma = \dfrac{5}{2}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \dfrac{4}{2} = 2$

$\alpha\beta\gamma = \dfrac{-6}{3} = -3$

(ii) sum of roots: $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} = \dfrac{(\alpha\beta)^2 + (\beta\gamma)^2 + (\alpha\gamma)^2}{(\alpha\beta\gamma)^2}$

$$= \frac{(\alpha\beta + \beta\gamma + \alpha\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)}{(\alpha\beta\gamma)^2}$$

$$= \frac{4 - 2(-3)(\frac{5}{2})}{9} = \frac{19}{9}$$

sum of products: $\dfrac{1}{\alpha^2}.\dfrac{1}{\beta^2} + \dfrac{1}{\alpha^2}.\dfrac{1}{\gamma^2} + \dfrac{1}{\beta^2}.\dfrac{1}{\gamma^2} = \dfrac{1}{(\alpha\beta)^2} + \dfrac{1}{(\alpha\gamma)^2} + \dfrac{1}{(\beta\gamma)^2}$

$$= \frac{\gamma^2 + \beta^2 + \alpha^2}{(\alpha\beta\gamma)^2}$$

$$= \frac{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)^2}{(\alpha\beta\gamma)^2}$$

$$= \frac{\frac{25}{4} - 2(2)}{9} = \frac{9}{36} = \frac{1}{4}$$

product of roots: $\frac{1}{\alpha^2}.\frac{1}{\beta^2}.\frac{1}{\gamma^2}=\frac{1}{(\alpha\beta\gamma)^2}=\frac{1}{9}$

therefore the equation is $x^3-\frac{19}{9}x^2+\frac{1}{4}x-\frac{1}{9}=0$

$$36x^3-76x^2+9x-4=0$$

3. (a) (i)

$$\sec^2\theta=\frac{\cos ec\theta}{\cos ec\theta-\sin\theta}$$
$$=\frac{1}{\sin\theta}\div\frac{1-\sin^2\theta}{\sin\theta}$$
$$=\frac{1}{1-\sin^2\theta}$$
$$=\frac{1}{\cos^2\theta}$$
$$=\sec^2\theta$$

(ii)

$$\frac{\cos ec\theta}{\cos ec\theta-\sin\theta}=\frac{4}{3}$$
$$\sec^2\theta=\frac{4}{3}$$
$$\sec\theta=\pm\frac{2}{\sqrt{3}}$$
$$\cos\theta=\pm\frac{\sqrt{3}}{2}$$
$$\cos^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{6}$$

$\cos\theta=\frac{\sqrt{3}}{2}\Rightarrow\theta=\frac{\pi}{6}$

$\theta=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$

$\cos\theta=-\frac{\sqrt{3}}{2}\Rightarrow\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

$\theta=\pi+\frac{\pi}{6}=\frac{7\pi}{6}$

(b) (i)

$$\sin\theta+\cos\theta=r\sin\theta\cos\alpha+r\cos\theta\sin\alpha$$
$$r\cos\alpha=1$$
$$r\sin\alpha=1$$
$$\tan\alpha=1$$
$$\therefore\alpha=\frac{\pi}{4}$$
$$r=\sqrt{1^2+1^2}=\sqrt{2}$$
$$\sin\theta+\cos\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$$

(ii) max value of $\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$ is $\sqrt{2}$

at max. value $\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)=1$

$$\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$
$$\theta+\frac{\pi}{4}=\frac{\pi}{2}$$
$$\theta=\frac{\pi}{4}$$

(c) $\tan(A+B+C)=\tan[(A+B)+C]$

$$=\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$$

$$=\frac{\frac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\left(\frac{\tan A+\tan B}{1-\tan A\tan B}\right)\tan C}$$

$$=\frac{\frac{\tan A+\tan B+\tan C(1-\tan A\tan B)}{1-\tan A\tan B}}{\frac{1-\tan A\tan B-(\tan A+\tan B)\tan C}{1-\tan A\tan B}}$$

$$=\frac{\tan A+\tan B+\tan C(1-\tan A\tan B)}{1-\tan A\tan B-(\tan A+\tan B)\tan C}$$

$$=\frac{\tan A+\tan B-\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}$$

4. (a) (i) $\sin\theta=x$

$\cos^2\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$ since $0<\theta<\frac{\pi}{2}$

$$\tan\theta=\frac{\sin\theta}{\cos\theta}$$

$$=\frac{x}{\sqrt{1-x^2}}$$

(ii) $y=\tan 2t=\dfrac{2\tan t}{1-\tan^2 t}$

$$=\frac{\frac{2x}{\sqrt{1-x^2}}}{1-\left(\frac{x}{\sqrt{1-x^2}}\right)^2}$$

$$=\frac{\frac{2x}{\sqrt{1-x^2}}}{1-\frac{x^2}{1-x^2}}$$

$$=\frac{\frac{2x}{\sqrt{1-x^2}}}{\frac{1-2x^2}{1-x^2}}$$

$$=\frac{2x\sqrt{1-x^2}}{1-2x^2}$$

(b) (i) $|\mathbf{u}| = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{14}$

$|\mathbf{v}| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{30}$

(ii) $\mathbf{u.v} = (1)(2) + (-3)(1) + (2)(5)$

$= 2 - 3 + 10 = 9$

$$\cos\theta = \frac{9}{\sqrt{14}\sqrt{30}}$$

(c) $d_1 = 2d_2$

$d_1 = \sqrt{x^2 + y^2}$

$d_2 = y$

$\sqrt{x^2 + y^2} = 2y$

$x^2 + y^2 = 4y^2$

$3y^2 - x^2 = 0$

(d) $x^2 + y^2 = 9$

$2x + y + 3 = 0 \Rightarrow y = -2x - 3$

$x^2 + (-2x-3)^2 = 9$

$x^2 + 4x^2 + 12x + 9 = 9$

$5x^2 + 12x = 0$

$x(5x + 12) = 0$

$x = 0, -\frac{12}{5}$

$x = 0 \Rightarrow y = -3$

$x = -\frac{12}{5} \Rightarrow y = \frac{9}{5}$

5. (a) let $u = x + 1$

$\frac{du}{dx} = 1 \Rightarrow du = dx$

$\therefore \int (x+1)^{\frac{1}{3}} dx = \int u^{\frac{1}{3}} du$

$= \frac{3u^{\frac{4}{3}}}{4} + c$

$= \frac{3(x+1)^{\frac{4}{3}}}{4} + c$

(b) $y = x^3 - 1$

$y + 1 = x^3$

$x = (y+1)^{\frac{1}{3}}$

$x^2 = (y+1)^{\frac{2}{3}}$

$$V = \pi \int_{-1}^{0} (y+1)^{\frac{2}{3}} dy$$

$$= \pi \left[ \frac{3(y+1)^{\frac{5}{3}}}{5} \right]_{-1}^{0}$$

$$= \frac{\pi}{5}\left[3(0+1)^{\frac{5}{3}} - 3(-1+1)^{\frac{5}{3}}\right]$$
$$= \frac{\pi}{5}[3-0] = \frac{3\pi}{5} \text{ cubic units}$$

(c) $$\int_0^1 \frac{e^x}{e^x + e^{1-x}}\,dx = \int_0^1 \frac{e^{1-x}}{e^{1-x} + e^x}\,dx$$
$$= \int_0^1 \frac{e^{1-x} + e^x - e^x}{e^{1-x} + e^x}\,dx$$
$$= \int_0^1 \frac{e^{1-x} + e^x}{e^{1-x} + e^x}\,dx + \int_0^1 \frac{-e^x}{e^{1-x} + e^x}\,dx$$
$$= \int_0^1 dx - \int_0^1 \frac{e^x}{e^{1-x} + e^x}\,dx$$
$$2\int_0^1 \frac{e^x}{e^{1-x} + e^x}\,dx = [x]_0^1 = 1$$
$$\int_0^1 \frac{e^x}{e^{1-x} + e^x}\,dx = \frac{1}{2}$$

(d) (i) $$\frac{dy}{dt} = 0.02y$$
$$\frac{1}{y}\,dy = 0.02dt$$
$$\int \frac{1}{y}\,dy = \int 0.02dt$$
$$\ln y = 0.02t + \ln A$$
$$y = e^{0.02t + \ln A}$$
$$y = e^{0.02t}.e^{\ln A} = Ae^{0.02t}$$
When $t = 0$, $y = 10000$

Therefore $A = 10000$

$$y = 10000e^{0.02t}$$

(ii) $$20000 = 10000e^{0.02t}$$
$$2 = e^{0.02t}$$
$$\ln 2 = 0.02t$$
$$t = \frac{\ln 2}{0.02} = 34.7$$

6. (a) $$f'(x) = 6x^2 + 10x - 1$$
$$f'(x) = 6(3)^2 + 10(3) - 1$$
$$= 83$$

$$f(3) = 2(3)^3 + 5(3)^2 - 3 + 12$$
$$= 108$$

$$y - 108 = 83(x-3)$$
$$y = 83x - 141$$

(b) (i) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 2x + 3) = 3$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (ax + b) = b$

(ii) $f(x)$ is continuous at $x = 0$ if $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$f(0) = 0^2 + 2(0) + 3 = 3$

$\Rightarrow b = 3$

$a$ can take any real value

(iii) $\lim_{h \to 0^-} \frac{(0+h)^2 + 2(0+h) + 3 - [0^2 + 2(0) + 3]}{h}$

$\lim_{h \to 0^-} \frac{h^2 + 2h}{h}$

$\lim_{h \to 0^-} (h + 2) = 2$

$\lim_{h \to 0^+} \frac{a(0+h) + 3 - [a(0) + 3]}{h}$

$\lim_{x \to 0^+} \frac{ah}{h} = a$

hence $a = 2$ when $b = 3$

(c) $f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$

$= \frac{1}{2\sqrt{x}}$