Cape Unit 1 Paper 2 2014

1. (a)

p	q	r	$p \to q$	$r \to q$	$(p \to q) \wedge (r \to q)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	T	T	T

(b) (i)

$$y \oplus x = y^3 + x^3 + ay^2 + ax^2 - 5y - 5x + 16$$
$$x \oplus y = x^3 + y^3 + ax^2 + ay^2 - 5x - 5y + 16$$
$$y \oplus x = x \oplus y$$

$\therefore \oplus$ is commutative

(ii)

$$2 \oplus x = 2^3 + x^3 + 4a + ax^2 - 10 - 5x + 16$$
$$= x^3 + ax^2 - 5x + 14 + 4a$$

$$f(1) = 1 + a - 5 + 14 + 4a = 0$$
$$a = -2$$

$$f(x) = x^3 - 2x^2 - 5x + 6$$

$$
\begin{array}{r}
x^2 - x - 6 \\
x - 1 \overline{\big) x^3 - 2x^2 - 5x + 6} \\
\underline{x^3 - x^2} \\
-x^2 - 5x \\
\underline{-x^2 + x} \\
-6x + 6 \\
\underline{-6x + 6}
\end{array}
$$

$$x^2 - x - 6 = (x-3)(x+2)$$

$$\therefore f(x) = (x-1)(x-3)(x+2)$$

(c) let P_n be $\sum_{r=1}^{n} (2r-1)^2 = \frac{n}{3}(4n^2 - 1)$

for $n = 1$: LHS = 1 RHS= $\frac{1}{3}(4-1) = 1$

$\therefore$ true for $n = 1$

Assume that P_n is true when $n = k$

$$\sum_{r=1}^{k} (2r-1)^2 = \frac{k}{3}(4k^2 - 1)$$

Prove that P_n is true when $n = k+1$, that is,

$$\sum_{r=1}^{k+1} (2r-1)^2 = \frac{k+1}{3}\left(4(k+1)^2 - 1\right)$$

LHS

$$\sum_{r=1}^{k+1} (2r-1)^2 = \sum_{r=1}^{k} (2r-1)^2 + (k+1)^{th} \text{ term}$$
$$= \frac{k}{3}(4k^2 - 1) + [2(k+1) - 1]^2$$
$$= \frac{k}{3}(4k^2 - 1) + (2k+1)^2$$

RHS

$$\frac{k+1}{3}\left(4(k^2 + 2k + 1) - 1\right)$$
$$\frac{k+1}{3}\left[4k^2 + 8k + 3\right]$$
$$\frac{4k^3 + 8k^2 + 3k + 4k^2 + 8k + 3}{3}$$

$$= \frac{4k^3 - k + 3(4k^2 + 4k + 1)}{3} \qquad = \frac{4k^3 + 12k^2 + 11k + 3}{3}$$

$$= \frac{4k^3 - k + 12k^2 + 12k + 3}{3}$$

$$= \frac{4k^3 + 12k^2 + 11k + 3}{3}$$

Since LHS=RHS P_n is true for $n = k+1$. Therefore P_n is true for all $n \in N$ by the PMI.

2. (a) (i) (a) $f^2(x) = ff(x) = f(2x^2+1)$

$$= 2(2x^2+1)^2 + 1$$
$$= 2(4x^4 + 4x^2 + 1) + 1$$
$$= 8x^4 + 8x^2 + 3$$

(b) $f[g(x)] = f\left(\sqrt{\frac{x-1}{2}}\right)$

$$= 2\left(\sqrt{\frac{x-1}{2}}\right)^2 + 1$$
$$= 2\left(\frac{x-1}{2}\right) + 1$$
$$= x - 1 + 1$$
$$= x$$

(ii) since $f[g(x)] = x$, f and g are inverse functions

(b) $3\log\left(\frac{a+b}{2}\right) = \log a + 2\log b$

$$3\log\left(\frac{a+b}{2}\right) = \log\left(\frac{a+b}{2}\right)^3$$
$$= \log\left(\frac{a^3 + 3a^2b + 3ab^2 + b^3}{8}\right)$$
$$= \log\left(\frac{5ab^2 + 3ab^2}{8}\right) = \log\left(\frac{8ab^2}{8}\right)$$
$$= \log(ab^2)$$
$$= \log a + \log b^2$$
$$= \log a + 2\log b$$

OR

$$3\log\left(\frac{a+b}{2}\right) = \log a + 2\log b$$
$$\log\left(\frac{a+b}{2}\right)^3 = \log(ab^2)$$
$$\frac{a^3 + 3a^2b + 3ab^2 + b^3}{8} = ab^2$$
$$a^3 + 3a^2b + 3ab^2 + b^3 = 8ab^2$$
$$a^3 + 3a^2b + b^3 = 5ab^2$$

(c) (i) $e^x + \frac{1}{e^x} - 2 = 0$

$e^{2x} + 1 - 2e^x = 0$

$e^{2x} - 2e^x + 1 = 0$

$(e^x - 1)^2 = 0$

$e^x = 1$

$x = 0$

(ii) $\log_2(x+1) - \log_2(3x+1) = 2$

$\log_2\left(\frac{x+1}{3x+1}\right) = 2$

$\frac{x+1}{3x+1} = 4$

$x + 1 = 12x + 4$

$-11x = 3$

$x = -\frac{3}{11}$

(d) $\frac{\sqrt{3}-1}{\sqrt{3}+1} + \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{2}-1}{\sqrt{2}+1} + \frac{\sqrt{2}+1}{\sqrt{2}-1} = 10$

$\frac{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} + \frac{(\sqrt{2}-1)^2 + (\sqrt{2}+1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}$

$\frac{3 - 2\sqrt{3} + 1 + 3 + 2\sqrt{3} + 1}{3-1} + \frac{2 - 2\sqrt{2} + 1 + 2 + 2\sqrt{2} + 1}{2-1}$

$\frac{8}{2} + \frac{6}{1} = 10$

3. (a) (i) $\frac{\cot y - \cot x}{\cot x + \cot y} = \frac{\frac{\cos y}{\sin y} - \frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x} + \frac{\cos y}{\sin y}}$

$= \frac{\frac{\sin x \cos y - \cos x \sin y}{\sin x \sin y}}{\frac{\sin y \cos x + \cos y \sin x}{\sin x \sin y}}$

$= \frac{\sin x \cos y - \cos x \sin y}{\sin y \cos x + \cos y \sin x}$

$= \frac{\sin(x-y)}{\sin(x+y)}$

(ii) $\frac{\cot y - \cot x}{\cot x + \cot y} = \frac{\sin x \cos y - \cos x \sin y}{\sin y \cos x + \cos y \sin x}$

$\sin x = \frac{1}{2} \Rightarrow \cos x = \sqrt{1 - \sin^2 x}$

$= \sqrt{1 - \frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$

But $0 \le x \le \frac{\pi}{2}$, hence $\cos x = \frac{\sqrt{3}}{2}$

$\therefore \frac{\frac{1}{2}\cos y - \frac{\sqrt{3}}{2}\sin y}{\frac{\sqrt{3}}{2}\sin y + \frac{1}{2}\cos y} = 1$

$\cos y - \sqrt{3}\sin y = \sqrt{3}\sin y + \cos y$

$0 = 2\sqrt{3}\sin y$
$\sin y = 0$
$y = 0, \pi, 2\pi$

OR

$\dfrac{\cot y - \cot x}{\cot x + \cot y} = \dfrac{\sin(x-y)}{\sin(x+y)} = 1$
$\sin(x-y) - \sin(x+y) = 0$
$\sin x\cos y - \cos x\sin y - \sin x\cos y - \cos x\sin y = 0$
$-2\cos x\sin y = 0$
$2\cos x\sin y = 0$

But $\sin x = \frac{1}{2} \quad 0 \le x \le \frac{\pi}{2}$
$\cos x = +\sqrt{1-\frac{1}{4}} \ne 0$
hence $2\cos x\sin y = 0 \Rightarrow \sin y = 0$
$y = 0, \pi, 2\pi$

(b) (i) $3\sin 2\theta + 4\cos 2\theta = r\sin 2\theta\cos\alpha + r\cos 2\theta\sin\alpha$
$r\cos\alpha = 3$
$r\sin\alpha = 4$
$\tan\alpha = \frac{4}{3}$
$\alpha = \tan^{-1}\frac{4}{3} = 0.927$
$r = \sqrt{3^2+4^2} = 5$

$\therefore 3\sin 2\theta + 4\cos 2\theta = 5\sin(2\theta + 0.927)$

(ii) (a) at minimum $\sin(2\theta + 0.927) = -1$
$2\theta + 0.927 = \frac{3\pi}{2}$
$\theta = \dfrac{\frac{3\pi}{2} - 0.927}{2}$
$\theta = 1.89^c$

(b) max value of $f(\theta)$ is 5
min value of $f(\theta)$ is -5

max value of $\dfrac{1}{7-f(\theta)} = \dfrac{1}{7-5} = \dfrac{1}{2}$
min value of $\dfrac{1}{7-f(\theta)} = \dfrac{1}{7+5} = \dfrac{1}{12}$

4. (a) (i) $x - y = -1 \ldots (1)$
$x + y = 5 \ldots (2)$
$(1)+(2) \Rightarrow 2x = 4$
$x = 2$

$2 + y = 5$
$y = 3$

(ii) $(2, 3) = \left(\dfrac{x_1+1}{2}, \dfrac{y_1+2}{2}\right)$
$\dfrac{x_1+1}{2} = 2 \Rightarrow x_1 + 1 = 4 \Rightarrow x_1 = 3$

$$\frac{y_1+2}{2}=3 \Rightarrow y_1+2=6 \Rightarrow y_1=4$$
$$B(3,4)$$

(iii) the locus of p is a circle with centre $(2, 3)$ and radius $\sqrt{2}$ units.

(b) $y=\frac{t}{1-t^2}=\frac{t}{(1-t)(1+t)}$

$$=\frac{xt}{1-t}$$

$$y(1-t)=xt$$
$$y-ty=xt$$
$$y=(x+y)t$$
$$\frac{y}{x+y}=t$$

$$x=\frac{1}{1+t}=\frac{1}{1+\left(\frac{y}{x+y}\right)}=\frac{x+y}{x+2y}$$

$$x^2+2xy-x-y=0$$

(c) (i) $\vec{PQ}=\vec{OQ}-\vec{OP}$

$$=\begin{pmatrix}-1\\ \lambda\\ 5\end{pmatrix}-\begin{pmatrix}3\\ -2\\ 1\end{pmatrix}=\begin{pmatrix}-4\\ \lambda+2\\ 4\end{pmatrix}$$

$$= -4\mathbf{i}+(\lambda+2)\mathbf{j}+4\mathbf{k}$$

$$\vec{QR}=\begin{pmatrix}2\\ 1\\ -4\end{pmatrix}-\begin{pmatrix}-1\\ \lambda\\ 5\end{pmatrix}=\begin{pmatrix}3\\ 1-\lambda\\ -9\end{pmatrix}$$

$$=3\mathbf{i}+(1-\lambda)\mathbf{j}-9\mathbf{k}$$

$$\vec{RP}=\begin{pmatrix}3\\ -2\\ 1\end{pmatrix}-\begin{pmatrix}2\\ 1\\ -4\end{pmatrix}=\begin{pmatrix}1\\ -3\\ 5\end{pmatrix}$$

$$=\mathbf{i}-3\mathbf{j}+5\mathbf{k}$$

(ii) since $\vec{PQ}$ is the hypotenuse, $\vec{PR}$ is perpendicular to $\vec{QR}$

$$\vec{PR}.\vec{QR}=0$$
$$(\mathbf{i}-3\mathbf{j}+5\mathbf{k}).(3\mathbf{i}+(1-\lambda)\mathbf{j}-9\mathbf{k})=0$$
$$-3+3(1-\lambda)+45=0$$
$$-3\lambda=-45$$
$$\lambda=15$$

5. (a) (i) $\lim_{x\to 3-} f(x)=3a+2 \qquad \lim_{x\to 3+} f(x)=9a$

since $f(x)$ is continuous at $x=3$

$$\lim_{x\to 3-} f(x)=\lim_{x\to 3+} f(x)$$
$$3a+2=9a$$
$$a=\tfrac{1}{3}$$

(ii) $$\lim_{x\to 1} 2g(x) = 2\left(\frac{1+2}{b+1+4}\right)$$
$$= 2\left(\frac{3}{b+5}\right)$$

$$\lim_{x\to 0} g(x) = \frac{2}{4} = \frac{1}{2}$$

$$\therefore 2\left(\frac{3}{b+5}\right) = \frac{1}{2}$$
$$12 = b+5$$
$$7 = b$$

(b) $$f'(x) = \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}\right\}$$
$$= \lim_{h\to 0}\frac{\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\left(\sqrt{x}\sqrt{x+h}\right)}$$
$$= \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\left(\sqrt{x}\sqrt{x+h}\right)} \times \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}$$
$$= \lim_{h\to 0}\frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}\sqrt{x}+h\sqrt{x}\sqrt{x+h}\sqrt{x+h}}$$
$$= \lim_{h\to 0}\frac{-h}{h\left(x\sqrt{x+h}+\sqrt{x}(x+h)\right)}$$
$$= \frac{-1}{x\sqrt{x}+x\sqrt{x}} = \frac{-1}{2x\sqrt{x}} = \frac{-1}{2x^{\frac{3}{2}}}$$

(ii) let $u = x \quad \frac{du}{dx} = 1$
$$v = (1+x)^{\frac{1}{2}} \quad \frac{dv}{dx} = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{(1+x)^{\frac{1}{2}} - \frac{1}{2}x(1+x)^{-\frac{1}{2}}}{1+x}$$
$$= \frac{2(1+x)^{\frac{1}{2}} - x(1+x)^{-\frac{1}{2}}}{2(1+x)} = \frac{2(1+x)^{\frac{1}{2}} - \frac{x}{(1+x)^{\frac{1}{2}}}}{2(1+x)}$$
$$= \frac{\frac{2(1+x)-x}{(1+x)^{\frac{1}{2}}}}{2(1+x)}$$
$$= \frac{2+2x-x}{2(1+x)^{\frac{3}{2}}} = \frac{2+x}{2(1+x)^{\frac{3}{2}}}$$

(c) $x = \cos\theta$ $\quad \frac{dx}{d\theta} = -\sin\theta$

$y = \sin\theta$ $\quad \frac{dy}{d\theta} = \cos\theta$

$$\frac{dy}{dx} = \frac{\cos\theta}{-\sin\theta}$$
$$= -\cot\theta$$

6. (a) (i) (a)

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$
$$y = x^3 - 2x^2 + x + c$$
$$-4 = (-1)^3 - 2(-1)^2 - 1 + c$$
$$0 = c$$
$$\therefore y = x^3 - 2x^2 + x$$

(b)

$$3x^2 - 4x + 1 = 0$$
$$(3x-1)(x-1) = 0$$
$$x = \tfrac{1}{3}, 1$$
$$x = 1, y = 0$$
$$x = \tfrac{1}{3}, y = \tfrac{4}{27}$$

Stationary points $(1, 0)$ and $\left(\tfrac{1}{3}, \tfrac{4}{27}\right)$

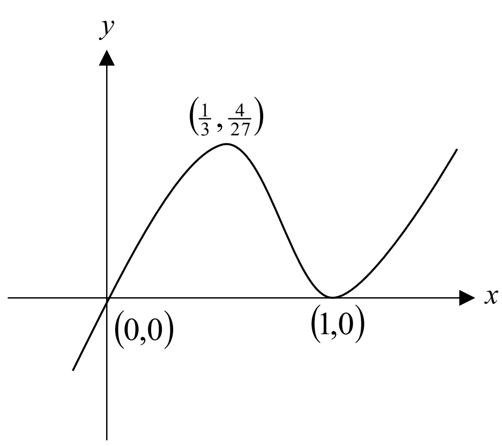
$$\frac{d^2y}{dx^2} = 6x - 4$$

at $x = 1$, $\frac{d^2y}{dx^2} = 2 > 0$ min point

at $x = \tfrac{1}{3}$, $\frac{d^2y}{dx^2} = -2 < 0$ max point

(ii)

$$y = x^3 - 2x^2 + x$$
$$0 = x(x^2 - 2x + 1)$$
$$0 = x(x-1)^2$$
$$x = 0, 1$$



(b) (i) let $u = 1 + x^2$

$$\frac{du}{dx} = 2x$$
$$du = 2x\,dx$$
$$x = 0 \Rightarrow u = 1$$
$$x = 3 \Rightarrow u = 10$$

$$\int_0^3 f(x)dx = \int_1^{10} u^{\frac{1}{2}} du$$

$$= \frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{10}$$

$$= \frac{2}{3}\left(10^{\frac{3}{2}} - 1\right) = 20.4$$

(ii) $$V = \pi \int_0^2 y^2 dx = \pi \int_0^2 4x^2\left(1 + x^2\right)dx$$

$$= 4\pi \int_0^2 \left(x^2 + x^4\right)dx$$

$$= 4\pi \left[\frac{x^3}{3} + \frac{x^5}{5}\right]_0^2$$

$$= 4\pi\left(\frac{8}{3} + \frac{32}{5}\right)$$

$$= \frac{544\pi}{15} \text{ cubic units}$$