CAPE Pure Maths UNIT 2 (2012)

QUESTION 1

(a) (i) a) $y = x^2 e^x$

$$\frac{dy}{dx} = 2xe^x + x^2e^x = e^x(x^2 + 2x)$$

$$\frac{d^2y}{dx^2} = e^x(x^2 + 2x) + e^x(2x + 2)$$

$$= e^x(x^2 + 4x + 2)$$

b) $\frac{dy}{dx} = 0$

$$e^x(x^2 + 2x) = 0$$

$$x^2 + 2x = 0$$

$$x(x + 2) = 0$$

$$x = -2, 0$$

c) $\frac{d^2y}{dx^2} = 0$

$$e^x(x^2 + 4x + 2) = 0$$

$$x^2 + 4x + 2 = 0$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-4 \pm 2\sqrt{2}}{2}$$

$$x = -2 \pm \sqrt{2}$$

(ii) when $x = -2$

$$\frac{d^2y}{dx^2} = e^{-2}((-2)^2 + 4(-2) + 2)$$

$$= -\frac{2}{e^2}$$

Maximum point

When $x = 0$

$$\frac{d^2y}{dx^2} = e^0(0^2 + 4(0) + 2) = 2$$

Minimum point

When $x = -2 \pm \sqrt{2}$

Points of inflection

(b) (i) $x = \sin^{-1}\sqrt{t}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1-t}} \times \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}\sqrt{1-t}}$$

$$y = t^2 - 2t$$

$$\frac{dy}{dt} = 2t - 2$$

$$\frac{dy}{dx} = (2t - 2)2\sqrt{t(1-t)}$$

$$= 4(t-1)\sqrt{t(1-t)}$$

(ii) when $t = \frac{1}{2}$

$$x = \sin^{-1}\sqrt{\frac{1}{2}} = \frac{\pi}{4}$$

$$y = \left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) = -\frac{3}{4}$$

$$\frac{dy}{dx} == 4\left(\frac{1}{2} - 1\right)\sqrt{\frac{1}{2}\left(1 - \frac{1}{2}\right)} = -1$$

$$y = mx + c \quad m = -1, \left(\frac{\pi}{4}, -\frac{3}{4}\right)$$

$$-\frac{3}{4} = -1\left(\frac{\pi}{4}\right) + c$$

$$\frac{\pi - 3}{4} = c$$

$$y = -x + \frac{\pi - 3}{4}$$

QUESTION 2

(a) (i) $\frac{x^2-3x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$x^2 - 3x = A(x^2+1) + (Bx+C)(x-1)$

When $x = 1$

$-2 = 2A$

$-1 = A$

Equating coefficients of x^2:

$1 = A + B$

$2 = B$

Equating coefficients of x:

$-3 = -B + C$

$-1 = C$

$$\frac{x^2-3x}{(x-1)(x^2+1)} = \frac{2x-1}{x^2+1} - \frac{1}{x-1}$$

(ii) $\int \frac{x^2-3x}{x^3-x^2+x-1}\,dx$

$$= \int \frac{2x-1}{x^2+1}\,dx - \int \frac{1}{x-1}\,dx$$

$$= \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx - \int \frac{1}{x-1}\,dx$$

$$= \ln(x^2+1) - \tan^{-1} x - \ln(x-1) + c$$

(b) (i) $\sin 2x = \sin(3x - x)$

$\sin 2x = \sin 3x \cos x - \cos 3x \sin x$

$\cos 3x \sin x = \sin 3x \cos x - \sin 2x$

(ii) $I_m = \int \cos^m x \sin 3x \; dx$ and $J_m = \int \cos^m x \sin 2x \; dx$

$u = \cos^m x$

$du = m \cos^{m-1} x\,(-\sin x)$

$\quad = -m \cos^{m-1} x \sin x$

$dv = \sin 3x$

$v = -\frac{1}{3}\cos 3x$

$$\int \cos^m x \sin 3x \; dx = -\frac{1}{3}\cos 3x \cos^m x - \frac{m}{3}\int \cos 3x \cos^{m-1} x \sin x \; dx$$

$$I_m = -\frac{1}{3}\cos 3x \cos^m x - \frac{m}{3}\int (\sin 3x \cos x - \sin 2x)\cos^{m-1} x \; dx$$

$$I_m = -\frac{1}{3}\cos 3x \cos^m x - \frac{m}{3}\int \sin 3x \cos^m x \; dx + \frac{m}{3}\int \sin 2x \cos^{m-1} x \; dx$$

$$I_m = -\frac{1}{3}\cos 3x \cos^m x - \frac{m}{3} I_m + \frac{m}{3} J_{m-1}$$

$3I_m = -\cos 3x \cos^m x - mI_m + mJ_{m-1}$

$(m+3)I_m = mJ_{m-1} - \cos 3x \cos^m x$

(iii) $\cos x \sin 3x = \cos 3x \sin x + \sin 2x$

$$I_1 = \int \cos x \sin 3x \; dx$$

When $m = 1$

$4I_1 = J_0 - \cos 3x \cos x$

$$= \int_0^{\frac{\pi}{4}} \cos^0 x \sin 2x \, dx - \cos 3x \cos x$$

$$= \int_0^{\frac{\pi}{4}} \sin 2x \, dx - \left[\cos 3\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{4}\right) - \cos 3(0) \cos 0\right]$$

$$= \int_0^{\frac{\pi}{4}} \sin 2x \, dx - \left[-\frac{1}{2} - 1\right]$$

$$= \int_0^{\frac{\pi}{4}} \sin 2x \, dx + \frac{3}{2}$$

(iv) $\int_0^{\frac{\pi}{4}} \sin 2x \, dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}$

$$= \left[-\frac{1}{2}\cos\left(2\left(\frac{\pi}{4}\right)\right)\right] - \left[-\frac{1}{2}\cos(2(0))\right]$$

$$= \frac{1}{2}$$

QUESTION 3

(a) (i) $u_6 = u_1 r^5 = 486$

$u_{11} = u_1 r^{10} = 118\,098$

$\frac{u_1 r^{10}}{u_1 r^5} = r^5 = \frac{118098}{486} = 243$

$r^5 = 243$

$r = 3$

$u_1(243) = 486$

$u_1 = 2$

(ii) $S_n = \frac{a(r^n - 1)}{r - 1}$

$177146 = \frac{2(3^n - 1)}{3 - 1}$

$177146 = 3^n - 1$

$177147 = 3^n$

$\ln 177147 = \ln 3^n$

$\ln 177147 = n \ln 3$

$\frac{\ln 177147}{\ln 3} = n$

$11 = n$

(i) $u_r = r(r+2)$

(ii) $P_n: \sum_{r=1}^{n} n(n+2) = \frac{1}{6}n(n+1)(2n+7)$

$P_1: 1(1+2) = \frac{1}{6}(1)(1+1)(2(1)+7)$

$3 = 3$

Therefore P_1 is true

Assume that P_n is true for $n = k$

$P_k: \sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7)$

$P_{k+1}: \sum_{r=1}^{k+1} r(r+2) = \frac{1}{6}(k+1)(k+2)(2k+9)$

Now, $P_{k+1} = P_k + (k+1)$ term

$P_{k+1} = \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$

$$= \frac{k(k+1)(2k+7) + 6(k+1)(k+3)}{6}$$

$$= \frac{(k+1)}{6}[2k^2 + 7k + 6k + 18]$$

$$= \frac{(k+1)(2k+9)(k+2)}{6}$$

Therefore P_{k+1} is true whenever P_k is true.

Hence by Mathematical Induction $\sum_{r=1}^{n} n(n+2) = \frac{1}{6}n(n+1)(2n+7)$ for all $n \in \mathbb{N}$.

(b) (i) From formula sheet

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\cos 2x = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24}$$

$$= 1 - \frac{4}{2}x^2 + \frac{16}{24}x^4$$

$$= 1 - 2x^2 + \frac{2}{3}x^4$$

(ii) $\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$= \frac{1}{2} - \frac{1}{2}\left(1 - 2x^2 + \frac{2}{3}x^4\right)$$

$$= \frac{1}{2} - \frac{1}{2} + x^2 - \frac{1}{3}x^4$$

$$= x^2 - \frac{1}{3}x^4$$

QUESTION 4

(a) (i) $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

(ii) $\binom{n}{n-r} = \frac{n!}{(n-(n-r))!(n-r)!}$

$$= \frac{n!}{r!\,(n-r)!}$$

$$= \binom{n}{r}$$

(iii) $\left(x^2 - \frac{3}{x}\right)^8$

Coefficient of x^4

$\binom{8}{4}(x^2)^4\left(-\frac{3}{x}\right)^4$

$70x^8\frac{81}{x^4}$

$= 5670$

(iv) $(1+x)^{2n} = (1+x)^n(1+x)^n$

$$\binom{2n}{n}x^n = \left[\binom{n}{0}x^0 + \binom{n}{1}x^1 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right]\left[\binom{n}{n}x^n + \binom{n}{n-1}x^{n-1} + \cdots + \binom{n}{1}x^1 + \binom{n}{0}x^0\right]$$

$$= \left[\binom{n}{0}^2 x^n + \binom{n}{1}^2 x^n + \cdots + \binom{n}{n-1}^2 x^n + \binom{n}{n}x^n\right]$$

$$\binom{2n}{n} = c_0^2 + c_1^2 + c_2^2 + \cdots + c_{n-1}^2 + c_n^2$$

(b) (i) $f(x) = 2x^3 + 3x^2 - 4x - 1$

$f(0.2) = 2(0.2)^3 + 3(0.2)^2 - 4(0.2) - 1$
$= -\frac{208}{125}$

$f(2) = 2(2)^3 + 3(2)^2 - 4(2) - 1 = 19$

$f(x)$ is continuous on the interval [0.2, 2] and $f(0.2)f(2) < 0$. By the Intermediate Value Theorem there is some c such that $f(c)) = 0$. Therefore there is a root in the interval [0.2, 2].

(ii) $f(x) = 2x^3 + 3x^2 - 4x - 1$

$f'(x) = 6x^2 + 6x - 4$

$$x_2 = 0.6 - \frac{2(0.6)^3 + 3(0.6)^2 - 4(0.6) - 1}{6(0.6)^2 + 6(0.6) - 4}$$

$x_2 = 1.672727$

$x_3 = 1.231807$

$x_4 = 1.042692$

$x_5 = 1.00$

QUESTION 5

(a) (i) $4 \times 6P3 = 480$

(ii) $4 \times 7^3 = 1372$

(b) (i) $\frac{6C5}{11C5} = \frac{1}{77}$

(ii) 0 T, O G, 5 J = $6C5 = 6$

1 T, 1 G, 3 J = $2C1 \times 3C1 \times 6C3 = 120$

2 T, 2 G, 1 J = $2C2 \times 3C2 \times 6C1 = 18$

Total number of committees is $6 + 120 + 18 = 144$

(c) $B = A^2 - 3A - 1$

$$= \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{pmatrix}$$

(ii) $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{pmatrix} = -9I$

(iii) $AB = -9I$

$$A\left(-\tfrac{1}{9}B\right) = I$$

$$A^{-1} = -\frac{1}{9}B = -\frac{1}{9}\begin{pmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{pmatrix}$$

(iv) $B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = B^{-1}\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\tfrac{1}{9}\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} 9 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$$

QUESTION 6

(a) (i) $A = \frac{1+i}{1-i}$

$$= \frac{(1+i)(1+i)}{(1-i)(1+i)}$$
$$= \frac{1+2i+i^2}{1+1}$$
$$= \frac{2i}{2}$$
$$= i$$

$$B = \frac{\sqrt{2}}{1-i}$$
$$= \frac{\sqrt{2}(1+i)}{(1-i)(1+i)}$$
$$= \frac{\sqrt{2}(1+i)}{2}$$
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

(ii) $A + B = \frac{1+i}{1-i} + \frac{\sqrt{2}}{1-i} = \frac{1+\sqrt{2}+i}{1-i}$

$$A + B = i + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$$
$$= \frac{\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2} i$$
$$\arg(A+B) = \tan^{-1}\left(\frac{2+\sqrt{2}}{2} \div \frac{\sqrt{2}}{2}\right) = \frac{3\pi}{8}$$

(b) (i) $z^2 = i$

$(a+bi)^2 = i$

$a^2 - b^2 + 2abi = i$

Equating real parts

$a^2 - b^2 = 0$

$a^2 = b^2$ (1)

Equating imaginary parts

$2ab = 1$

$ab = \frac{1}{2}$

$a = \frac{1}{2b}$ (2)

Sub (2) into (1)

$\left(\frac{1}{2b}\right)2 = b^2$

$\frac{1}{4b^2} = b^2$

$\frac{1}{4} = b^4$

$\pm\frac{1}{\sqrt{2}} = b = \pm\frac{\sqrt{2}}{2}$

When $b = \frac{1}{\sqrt{2}}$

$$a = \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)} = \frac{\sqrt{2}}{2}$$

$$z_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

When $b = -\frac{\sqrt{2}}{2}$

$$a = \frac{1}{2\left(-\frac{\sqrt{2}}{2}\right)} = -\frac{\sqrt{2}}{2}$$

$$z_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$$

(ii) $z^2 - (3+5i)z - (4-7i) = 0$

$$z = \frac{3+5i \pm \sqrt{(3+5i)^2 - 4(1)(-4+7i)}}{2(1)}$$
$$= \frac{3+5i \pm \sqrt{9+30i-25+16-28i}}{2}$$
$$= \frac{3+5i \pm \sqrt{2i}}{2}$$
$$= \frac{3+5i \pm \sqrt{2}\sqrt{i}}{2}$$
$$= \frac{3+5i \pm \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)}{2}$$
$$= \frac{3+5i \pm (1+i)}{2}$$
$$z = \frac{3+5i+1+i}{2} = 2+3i$$
$$z = \frac{3+5i-(1+i)}{2} = 1+2i$$

(c) $\cos 6\theta + i \sin 6\theta = (\cos\theta + i \sin\theta)^6$

$= \cos^6\theta + 6\cos^5\theta\,(i\sin\theta) + 15\cos^4\theta\,(i\sin\theta)^2 + 20\cos^3\theta\,(i\sin\theta)^3 + 15\cos^2\theta\,(i\sin\theta)^4$

$+6\cos\theta\,(i\sin\theta)^5 + (i\sin\theta)^6$

$= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin\theta - 20i\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta$

Equating real parts

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$