CAPE UNIT 1 Paper 2 (2012) SOLUTIONS

QUESTION 1

(a) $f(x) = 2x^3 - px^2 + qx - 10$

(i) $f(1) = 0$

$2(1)^3 - p(1)^2 + q(1) - 10 = 0$

$p - q = -8$

$f(-1) = -6$

$2(-1)^3 - p(-1)^2 + q(-1) - 10 = -6$

$-2 - p - q - 10 = -6$

$p + q = -6$

$p - q = -8$

$p + q = -6$

$-2q = -2$

$q = 1$

$p = -7$

(ii) $f(x) = 2x^3 + 7x^2 + x - 10$

$f(x) = (x - 1)(2x^2 + 9x + 10)$

$f(x) = (x - 1)(2x + 5)(x + 2)$

(b) $\left(\sqrt{x} + \sqrt{y}\right)^2 = 16 + \sqrt{240}$

$x + y + 2\sqrt{xy} = 16 + \sqrt{240}$

$x + y = 16 \quad \rightarrow \quad x = 16 - y$

$2\sqrt{xy} = \sqrt{240}$

$2\sqrt{xy} = 2\sqrt{60}$

$xy = 60$

$(16 - y)y = 60$

$16y - y^2 - 60 = 0$

$y^2 - 16y + 60 = 0$

$(y - 10)(y - 6) = 0$

$y = 6, 10$

$x = 10, 6$

(c) (i) $|3x - 7| \leq 5$

$-5 \leq 3x - 7 \leq 5$

$2 \leq 3x \leq 12$

$\frac{2}{3} \leq x \leq 4$

(ii) $|3x - 7| + 5 \leq 0$

$|3x - 7| \leq -5$

There is no solution since modulus cannot be negative.

QUESTION 2

(a) (i) $f\big(f(x)\big)] = f(x^2-3)$

$= (x^2-3)^2 - 3$

$= x^4 - 6x^2 + 9 - 3$

$= x^4 - 6x^2 + 6$

(ii) $f\big(f(x)\big) = f(x+3)$

$x^4 - 6x^2 + 6 = (x+3)^2 - 3$

$x^4 - 6x^2 + 6 = x^2 + 6x + 6$

$x^4 - 7x^2 - 6x = 0$

$x(x^3 - 7x - 6) = 0$

$x(x+1)(x-3)(x+2) = 0$

By Synthetic Division or Long Division

$x = -2, -1, 0, 3$

(b) (i) $\alpha + \beta = \frac{3}{4}$

$\alpha\beta = \frac{1}{4}$

(ii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{3}{4}\right)^2 - 2\left(\frac{1}{4}\right)$

$= \frac{1}{16}$

(iii) $x^2 - \left(\frac{2}{\alpha^2} + \frac{2}{\beta^2}\right)x + \left(\frac{2}{\alpha^2}\right)\left(\frac{2}{\beta^2}\right) = 0$

$x^2 - \frac{2\beta^2 + 2\alpha^2}{\alpha^2\beta^2}x + \frac{4}{\alpha^2\beta^2} = 0$

$x^2 - \frac{2(\alpha^2+\beta^2)}{(\alpha\beta)^2}x + \frac{4}{(\alpha\beta)^2} = 0$

$x^2 - \frac{2\left(\frac{1}{16}\right)}{\left(\frac{1}{4}\right)^2}x + \frac{4}{\left(\frac{1}{4}\right)^2} = 0$

$x^2 - 2x + 64 = 0$

(c) (i) $\log_{10}\left(\frac{1}{3}\right) + \log_{10}\left(\frac{3}{5}\right) + \log_{10}\left(\frac{5}{7}\right) + \log_{10}\left(\frac{7}{9}\right) + \log_{10}\left(\frac{9}{10}\right)$

$= \log_{10}\left(\frac{1}{3}\times\frac{3}{5}\times\frac{5}{7}\times\frac{7}{9}\times\frac{9}{10}\right)$

$= \log_{10}\left(\frac{1}{10}\right)$

$= \log_{10} 10^{-1}$

$= -1$

(ii) $\sum_{r=1}^{99} \log_{10}\left(\frac{r}{r+1}\right)$

$= \log_{10}\left(\frac{1}{2}\right) + \log_{10}\left(\frac{2}{3}\right) + \log_{10}\left(\frac{3}{4}\right) + \cdots + \log_{10}\left(\frac{98}{99}\right) + \log_{10}\left(\frac{99}{100}\right)$

$= \log_{10}\left(\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\ldots\times\frac{98}{99}\times\frac{99}{100}\right)$

$= \log_{10} \frac{1}{100}$

$= \log_{10} 10^{-2}$

$= -2$

QUESTION 3

(a) (i) $\cos 3\theta$

$= \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos\theta\left[\cos^2\theta - \sin^2\theta - \frac{1}{2}\right]$

(ii) $\frac{1}{2}[\sin 6\theta - \sin 2\theta]$

$= \frac{1}{2}[2\cos\left(\frac{6\theta + 2\theta}{2}\right)\sin\left(\frac{6\theta - 2\theta}{2}\right)$

$= \cos 4\theta \sin 2\theta$

$= (2\cos^2 2\theta - 1)\sin 2\theta$

(iii) $(2\cos^2 2\theta - 1)\sin 2\theta = 0$

$\sin 2\theta = 0$

$2\theta = 0, \pi$

$2\cos^2 2\theta - 1 = 0$

$\cos^2 2\theta = \frac{1}{2}$

$\cos 2\theta = \pm\frac{1}{\sqrt{2}}$

$R.A = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$I:\ 2\theta = \frac{\pi}{4}$

$II: 2\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\theta = 0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{2}$

(b) $2\cot^2\theta + \cos\theta = 0$

$\frac{2\cos^2\theta}{\sin^2\theta} + \cos\theta = 0$

$2\cos^2\theta + \sin^2\theta\cos\theta = 0$

$2\cos^2\theta + (1 - \cos^2\theta)\cos\theta = 0$

$2\cos^2\theta + \cos\theta - \cos^3\theta = 0$

$\cos^3\theta - 2\cos^2\theta - \cos\theta = 0$

$\cos\theta\,(\cos^2\theta - 2\cos\theta - 1) = 0$

$\cos\theta = 0$

$\cos^2\theta - 2\cos\theta - 1 = 0$

$\cos\theta = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(1)}$

$\cos\theta = \frac{2 \pm \sqrt{12}}{2}$

$\cos\theta = 1 \pm \sqrt{3}$

QUESTION 4

(a) (i) $x = 3\tan\theta\,, y = 3\sec\theta$

$$\frac{x}{3} = \tan\theta$$

$$\frac{x^2}{9} = \tan^2\theta$$

$$y^2 = 9\sec^2\theta$$

$$\frac{y^2}{9} = \sec^2\theta$$

$$RECALL: \tan^2\theta + 1 = \sec^2\theta$$

$$\frac{x^2}{9} + 1 = \tan^2\theta + 1$$

$$\frac{y^2}{9} = \frac{x^2}{9} + 1$$

$$y^2 = x^2 + 9$$

(ii) $y^2 = \frac{x^2}{9} + 1$

$$y = \sqrt{10x}$$

$$10x = x^2 + 9$$

$$10x = x^2 + 9$$

$$x^2 - 10x + 9 = 0$$

$$(x-1)(x-9) = 0$$

$$x = 1, 9$$

$$y = \sqrt{10(1)} = \sqrt{10}$$

$$(1, \sqrt{10})$$

$$y = \sqrt{10(9)} = 3\sqrt{10}$$

$$(9, 3\sqrt{10})$$

(b) (i) $p = -3i + 4j, \quad q = -i + 6j$

(ii) $p - q = (-3i + 4j) — i + 6j = 2$

(iii) $p.q = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.\begin{pmatrix} -1 \\ 6 \end{pmatrix} = 3 + 24 = 27$

(iv) $p.q = |p||q|\cos\theta$

$$\theta = \cos^{-1}\left(\frac{27}{\sqrt{(-3)^2 + 4^2}\sqrt{(-1)^2 + 6^2}}\right) = 27.41°$$

QUESTION 5

(a) (i) $x^2 - 4 = 0$

$x = \pm 2$

(ii) $\lim\limits_{x \to -2} \frac{x^3+8}{x^2-4}$

$= \lim\limits_{x \to 0} \frac{(x+2)(x^2-2x+4)}{(x+2)(x-2)}$

$= \lim\limits_{x \to 0} \frac{x^2-2x+4}{x-2}$

$= \frac{(-2)^2 - 2(-2) + 4}{-2-2}$

$= -3$

(iii) $\lim\limits_{x \to 0} \frac{2x^3+4x}{\sin 2x}$

$= \lim\limits_{x \to 0} (2x^3 + 4x)\left(\frac{2x}{2x \sin 2x}\right)$

$= \lim\limits_{x \to 0} (x^2 + 2)\left(\frac{2x}{\sin 2x}\right)$

$= (0^2 + 2)(1)$

$= 2$

(b) (i) a) $\lim\limits_{x \to 1^+} x^2 + 1 = 1^2 + 1 = 2$

b) $2 = 4 + p(1)$

$p = -2$

(ii) $f(1) = 2$

(c) $M = ut^2 + \frac{v}{t^2}$

$-1 = u(1)^2 + \frac{v}{(1)^2}$

$u + v = -1 \qquad (1)$

$\frac{dM}{dt} = \frac{35}{4}$

$2ut - 2vt^{-3} = \frac{35}{4}$

$4u - \frac{v}{4} = \frac{35}{4}$

$16u - v = 35 \qquad (2)$

$u + v = -1 \qquad (1)$

$16u - v = 35 \qquad (2)$

$17u = 34$

$u = 2$

$v = -3$

QUESTION 6

(a) (i) $y = \sqrt{4x^2 - 7} = (4x^2 - 7)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(4x^2 - 7)^{-\frac{1}{2}}(8x) = 4x(4x^2 - 7)^{-\frac{1}{2}} = \frac{4x}{\sqrt{4x^2 - 7}}$$

$$y\frac{dy}{dx} = \sqrt{4x^2 - 7}\left(\frac{4x}{\sqrt{4x^2 - 7}}\right) = 4x$$

(ii) $\frac{d^2y}{dx^2} = 4(4x^2 - 7)^{-\frac{1}{2}} + 4x\left[-\frac{1}{2}(4x^2 - 7)^{-\frac{3}{2}}(8x)\right]$

$$= 4(4x^2 - 7)^{-\frac{1}{2}} - 16x^2(4x^2 - 7)^{-\frac{3}{2}}$$

$$= \frac{4}{\sqrt{4x^2 - 7}} - \frac{16x^2}{(4x^2 - 7)\sqrt{4x^2 - 7}}$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \sqrt{4x^2 - 7}\left(\frac{4}{\sqrt{4x^2 - 7}} - \frac{16x^2}{(4x^2 - 7)\sqrt{4x^2 - 7}}\right) + \left(\frac{4x}{\sqrt{4x^2 - 7}}\right)^2$$

$$= 4 - \frac{16x^2}{4x^2 - 7} + \frac{16x^2}{4x^2 - 7}$$

$$= 4$$

(b) (i) $\frac{dy}{dx} = 3x^2 - 6x$

$$y = \frac{3x^3}{3} - \frac{6x^2}{2} + c$$

$$y = x^3 - 3x^2 + c$$

$$0 = (-1)^3 - 3(-1)^2 + c$$

$$c = 4$$

$$y = x^3 - 3x^2 + 4$$

(ii) $3x^2 - 6x = 0$

$$3x(x - 2) = 0$$

$$x = 0, 3$$

$$y = 0^3 - 3(0)^2 + 4 = 4$$

$$(0, 4)$$

$$y = 2^3 - 3(2)^2 + 4 = 0$$

$$(2, 0)$$

$$\frac{d^2y}{dx^2} = 6x - 6$$

At $(0, 4)$:

$$\frac{d^2y}{dx^2} = 6(0) - 6 = -6$$

Maximum

At $(2, 0)$:

$$\frac{d^2y}{dx^2} = 6(2) - 6 = 6$$

Minimum

(iv) $y = 0$

$$x^3 - 3x^2 + 4 = 0$$

$$(x - 2)(x - 2)(x + 1) = 0$$

$$x = -1, 2$$

$$(-1, 0) \quad (2, 0)$$

(v)