SECTION A

Module 1

Answer BOTH questions.

1. (a) (i) Let p and q be any two propositions. Complete the truth table below.

p	q	$p \longrightarrow q$	$q \longrightarrow p$	$(p \longrightarrow q) \wedge (q \longrightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

[3 marks]

(ii) Hence, state whether the statements $q \longrightarrow p$ and $(p \longrightarrow q) \wedge (q \longrightarrow p)$ are logically equivalent. **Justify your response.**

NOT LOGICALLY EQUIVALENT.

THE TRUTH TABLES ARE NOT

IDENTICAL

[2 marks]

(b) Let x and y be negative real numbers and let z be any real number. Use a counter example to show that the statement "if $x > y$ then $xz > yz$" is false.

Let $x = -2$ and $y = -5$ and $z = -3$

$-2 > -5$ is true

$xz = (-2)(-3) = 6$

$yz = (-5)(-3) = 15$

since $6 < 15 \Rightarrow xz < yz$

so statement is false.

[3 marks]

GO ON TO THE NEXT PAGE

(c) The expression $f(x) = 6x^3 + px^2 + qx + 2$ is divisible by $2x - 1$ and has a remainder of 2 when divided by $x - 1$.

Calculate the values of p and q.

since $2x-1$ is a factor of $f(x)$

$f\left(\frac{1}{2}\right) = 0$

$f\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^3 + p\left(\frac{1}{2}\right)^2 + q\left(\frac{1}{2}\right) = 0$

$= \frac{1}{4}p + \frac{1}{2}q + \frac{11}{4} = 0$

$p + 2q + 11 = 0$

$f(1) = 2$

$= 6(1)^3 + p(1)^2 + q(1) + 2 = 2$

$= p + q + 8 = 2$

$\Rightarrow p + q = -6$

$p + 2q = -11$

$p + q = -6$

$q = -5$

$p - 5 = -6$

$p = -1$

$p = -1 \qquad q = -5$

[9 marks]

GO ON TO THE NEXT PAGE

(d) (i) Solve the logarithmic equation $\log_3 (x^2 - 9) - \log_3 (x + 3) = 3$.

$$\log_3 (x+3)(x-3) - \log_3 (x+3) = 3$$

$$\log_3 (x-3) = 3$$

$$x-3 = 27$$

$$x = 30$$

[4 marks]

(ii) Show that $\sqrt{320x^3} + \sqrt{125x^3}$ simplifies to $13x\sqrt{5x}$.

$$= \sqrt{320x^2 x} + \sqrt{125x^2 x}$$

$$= x\sqrt{320x} + x\sqrt{125x}$$

$$= x\sqrt{64 \times 5x} + x\sqrt{5 \times 25x}$$

$$= 8x\sqrt{5x} + 5x\sqrt{5x}$$

$$= (\sqrt{5x})(8x + 5x)$$

$$= (\sqrt{5x})(13x)$$

[4 marks]

Total 25 marks

2. (a) Let $f(x) = 7x + 2$. Prove that f is bijective.

Let x_1 and x_2

$f(x_1) = 7x_1 + 2$

$f(x_2) = 7x_2 + 2$

$7x_1 + 2 = 7x_2 + 2$

$7x_1 = 7x_2$

$x_1 = x_2$ so $f(x)$ is injective.

Let y be an element in the codomain

$y = 7x - 2$

$x = \frac{y+2}{7}$

$f\left(\frac{y+2}{7}\right) = 7\left(\frac{y+2}{7}\right) - 2 = y$

[5 marks]

So for every x in the domain there is a corresponding y in the codomain

$\therefore$ $f(x)$ is onto

Since f is both injective and onto f is bijective

(b) The roots of the cubic equation $3x^3 - x^2 - 2x + 1 = 0$ are α, β and γ. Determine the equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

$$3\left(\frac{1}{x}\right)^3 - \left(\frac{1}{x}\right)^2 - 2\left(\frac{1}{x}\right) + 1 = 0$$

$$\frac{3}{x^3} - \frac{1}{x^2} - \frac{2}{x} + 1 = 0$$

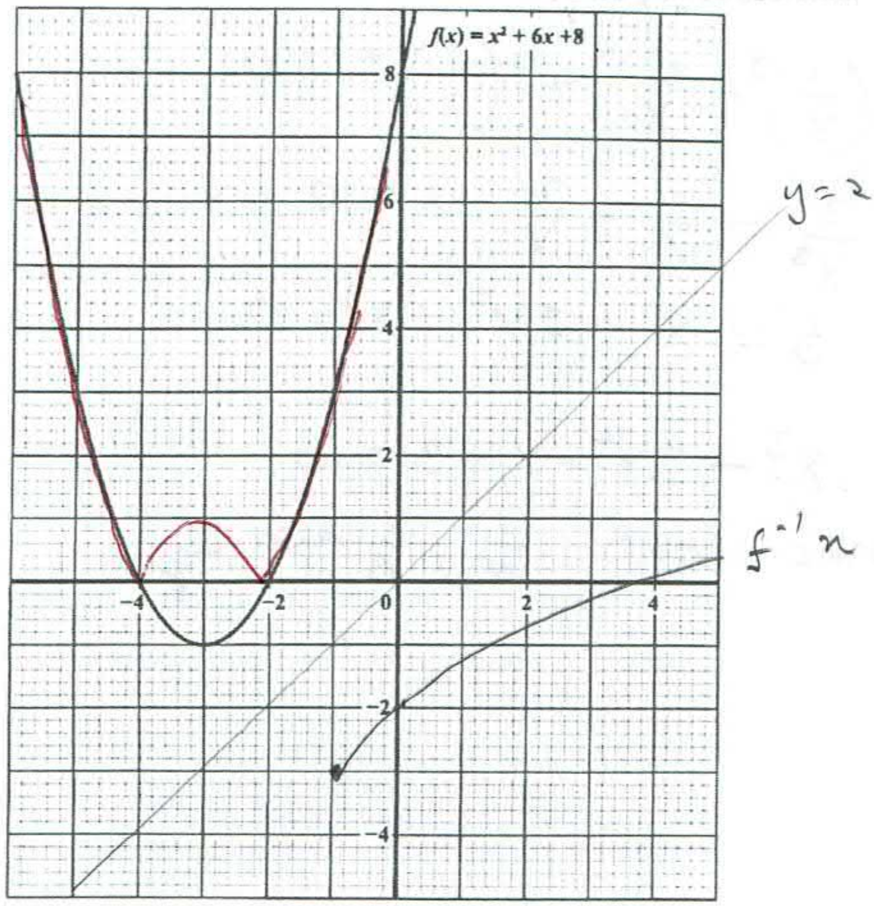
$$(\times x^3) \quad 3 - x - 2x^2 + x^3 = 0$$

so $x^3 - 2x^2 - x + 3 = 0$

has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$

[8 marks]

(c) The diagram below shows the graph of the curve $f(x) = x^2 + 6x + 8$.



(i) On the axes provided above, sketch and label the graph of $g(x) = |x^2 + 6x + 8|$. **[3 marks]**

(ii) On the same axes, sketch and label the inverse of f for $x \geq -3$. **[5 marks]**

(d) Given that $g(x) = \frac{2x+3}{x+3}$, prove that $g^{-1}(2)$ does not exist.

$x = \frac{2y+3}{y+3}$

$xy + 3x = 2y + 3$

$xy - 2y = 3 - 3x$

$y = \frac{3-3x}{x-2}$

$g^{-1}(x) = \frac{3-3x}{x-2} \quad x \neq 2$

$g^{-1}(2) =$ undefined

[4 marks]

Total 25 marks

SECTION B

Module 2

Answer BOTH questions.

3. (a) Prove that

$$\frac{1 - \sin\theta}{1 + \sin\theta} = (\sec\theta - \tan\theta)^2.$$

$$RHS = \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2$$

$$= \frac{(1 - \sin\theta)^2}{\cos^2\theta}$$

$$= \frac{(1 - \sin\theta)^2}{1 - \sin^2\theta}$$

$$= \frac{(1 - \sin\theta)^2}{(1 - \sin\theta)(1 + \sin\theta)}$$

$$= \frac{1 - \sin\theta}{1 + \sin\theta} = LHS$$

[9 marks]

(b) Solve the equation $2\cos^2 x - 3\sin x = 3$ for $0 \leq x \leq 2\pi$.

$2(1-\sin^2 x) - 3\sin x = 3$

$2 - 2\sin^2 x - 3\sin x - 3 = 0$

$-2\sin^2 x - 3\sin x - 1 = 0$

$2\sin^2 x + 3\sin x + 1 = 0$

$(2\sin x + 1)(\sin x + 1) = 0$

$\sin x = \frac{-1}{2}$ or $\sin x = -1$

$x = \left(\pi + \frac{\pi}{6}\right), \left(2\pi - \frac{\pi}{6}\right)$

$x = \frac{3\pi}{2}$

[9 marks]

GO ON TO THE NEXT PAGE

(c) Show that $\cos\left(\frac{\pi}{2}+x\right) = -\sin x$.

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos\left(\frac{\pi}{2}+x\right) = \cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x$$

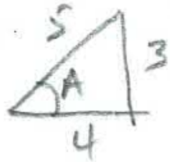
$$= 0 \cdot \cos x - 1 \cdot \sin x$$

$$= -\sin x.$$

[4 marks]

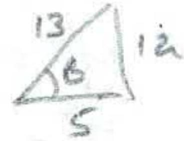
GO ON TO THE NEXT PAGE

(d) A and B are acute angles such that $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$. Calculate, without using tables or calculators, the EXACT value of $\cos(A - B)$.



$\sin A = \frac{3}{5}$

$\cos A = \frac{4}{5}$



$\cos B = 5/13$

$\sin B = 12/13$

$$\cos(A - B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{4}{5} \cdot \frac{5}{13} + \frac{3}{5} \cdot \frac{12}{13}$$

$$= \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$$

[3 marks]

Total 25 marks

4. (a) Obtain the Cartesian equation of the curve given in parametric form

$x = 3\cos t$ and $y = 4\sin t$.

$x^2 = 9\cos^2 t \quad y^2 = 16\sin^2 t$

$\frac{x^2}{9} = \cos^2 t \quad \frac{y^2}{16} = \sin^2 t$

$\cos^2 t + \sin^2 t = \frac{x^2}{9} + \frac{y^2}{16} = 1$

$\frac{x^2}{9} + \frac{y^2}{16} = 1$

[5 marks]

(b) The equation of a line is $x = 2 + t$, $y = 1 - 3t$ and $z = 4 + t$, and the equation of a plane is $x + 2y + z = 12$. Determine the point of intersection of the line and the plane.

$2 + t + 2(1 - 3t) + (4 + t) = 12$

$2 + t + 2 - 6t + 4 + t = 12$

$8 - 4t = 12$

$-4t = 4$

$t = -1$

Point of intersection

$(2 - 1), (1 + 3) (4 - 1)$

$(1, 4, 3)$

[6 marks]

GO ON TO THE NEXT PAGE

(c) (i) Determine the vector equation of the plane which passes through $(1, 5, -1)$ and which is perpendicular to the vector $\begin{pmatrix}2\\4\\3\end{pmatrix}$.

$$r \cdot \begin{pmatrix}2\\4\\3\end{pmatrix} = \begin{pmatrix}1\\5\\-1\end{pmatrix}\begin{pmatrix}2\\4\\3\end{pmatrix}$$

$$r \cdot \begin{pmatrix}2\\4\\3\end{pmatrix} = 2 + 20 - 3 = 19$$

[4 marks]

(ii) Hence, determine the coordinates of the point in the plane where $y = 3$ and $z = 1$.

$$2x + 4y + 3z - 19 = 0$$

$$2x + 12 + 3 - 19 = 0$$

$$2x - 4 = 0$$

$$x = 2$$

So point is $(2, 3, 1)$

[3 marks]

GO ON TO THE NEXT PAGE

DO NOT WRITE IN THIS AREA

(d) Given that a line is parallel to the vector $\mathbf{u} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and that the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is normal to the plane, calculate the angle between the line and the plane.

$$\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1-6+1 = -4$$

$$\sqrt{(1+9+1)}\sqrt{(1+4+1)} \cos\theta = -4$$

$$\sqrt{11} \cdot \sqrt{6}$$

$$\cos\theta = \frac{-4}{\sqrt{66}} =$$

$\theta = 119.5^\circ$ (angle between line and normal)

So angle between line and plane

$$119.5 - 90 = 29.5^\circ$$

[7 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

SECTION C

Module 3

Answer BOTH questions.

5. (a) Determine $\lim_{x \to \infty} \dfrac{2x^3 - 4x + 1}{3x^4 + x^2 - 2}$.

$\left(\div x^4\right)$

$= \lim_{x \to \infty} \dfrac{\frac{2}{x} - \frac{4}{x^3} + \frac{1}{x^4}}{3 + \frac{1}{x^2} - \frac{2}{x^4}}$

$= \dfrac{0 - 0 + 0}{3 + 0 - 0} = 0$

[5 marks]

GO ON TO THE NEXT PAGE

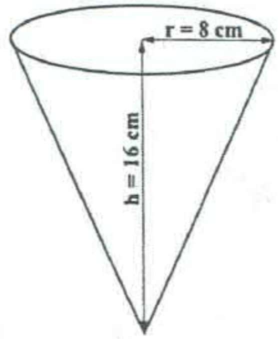
DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

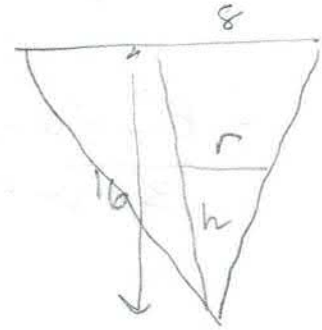
(b) A student goes to the water cooler to get a drink of water. The student has a cup in the shape of a cone. The water from the cooler is filling the cup at the rate of 12 cm^3 per second. If the height, h, of the cup is 16 cm and the radius, r, of its circular opening is 8 cm, how fast is the water in the cup rising when the height is 4 cm?

The volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius and h is the height.



$$\frac{dv}{dt} = 12$$

$$V = \frac{1}{3}\pi r^2 h$$



$$\frac{dh}{dt} = \frac{dv}{dt} \times \frac{dh}{dv}$$

$$\frac{dh}{dt} = 12 \times \frac{4}{\pi h^2}$$

when $h = 4$

$$\frac{dh}{dt} = 12 \times \frac{4}{\pi \times 16}$$

$$= \frac{3}{\pi} \text{ cm/s}$$

$$\frac{8}{16} = \frac{r}{h}$$

$h = 2r$ or $r = \frac{h}{2}$

$$V = \frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$$

$$= \frac{\pi h^3}{12}$$

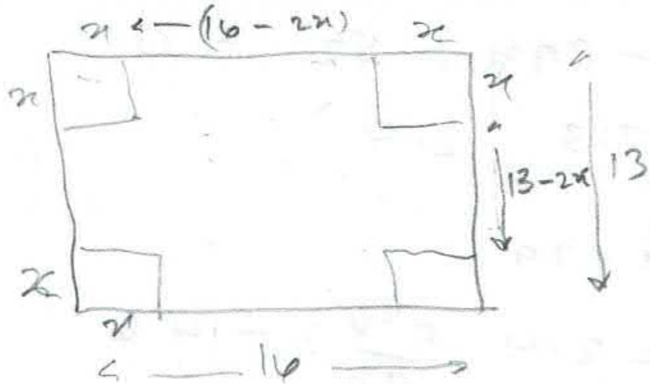
$$\frac{dv}{dh} = \frac{3\pi h^2}{12} = \frac{\pi h^2}{4}$$

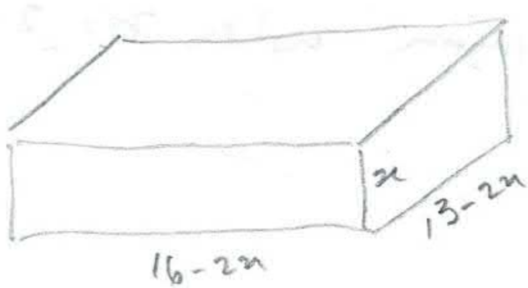
[8 marks]

GO ON TO THE NEXT PAGE

(c) A manufacturer of containers wants to make a container in the shape of a cuboid with an open top. The container is made from a flat sheet of metal of length 16 metres and width 13 metres. Squares of length x metres are cut from each corner of the sheet to create the sides of the container.

(i) Show that the volume of the container is $4x^3 - 58x^2 + 208x$.





[2 marks]

$$V = (16-2x)(13-2x)(x)$$

$$= (208 - 26x - 32x + 4x^2)x$$

$$V = 208x - 58x^2 + 4x^3$$

$V = 4x^3 - 58x^2 + 208x$

(ii) Using the method of the second derivative, determine the height, x, that will maximize the volume of the container.

$\frac{dV}{dx} = 12x^2 - 116x + 208 = 0$ for max volume

$3x^2 - 29x + 52 = 0$

$x = 7.3, \quad 2.4$

$\frac{d^2V}{dx^2} = 6x - 29$

when $x = 2.4$ $\frac{d^2V}{dx} = -14.6$

So V is maximum when $x = 2.4$

[9 marks]

(iii) Determine the maximum volume of the container.

$V = 4(2.4)^3 - 58(2.4)^2 + 208(2.4)$

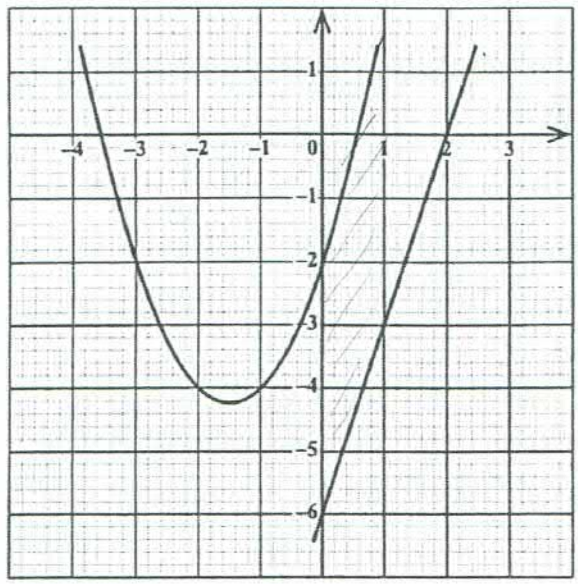
$= 220.416 \text{ cm}^2$

[1 mark]

Total 25 marks

GO ON TO THE NEXT PAGE

6. (a) Calculate the volume of the solid generated by revolving the region bounded by the line $y = 3x - 6$ and the parabola $y = x^2 + 3x - 2$, on the interval $[0, 1]$ about the x-axis.



$$V = \pi \int_0^1 \left| (x^2+3x-2)^2 - (3x-6)^2 \right| dx$$

$$= \pi \int_0^1 x^4 + 6x^3 + 5x^2 - 12x + 4 - (9x^2 - 36x + 36) \Big| dx$$

$$= \pi \int_0^1 \left| x^4 + 6x^3 - 4x^2 + 24x - 32 \right|$$

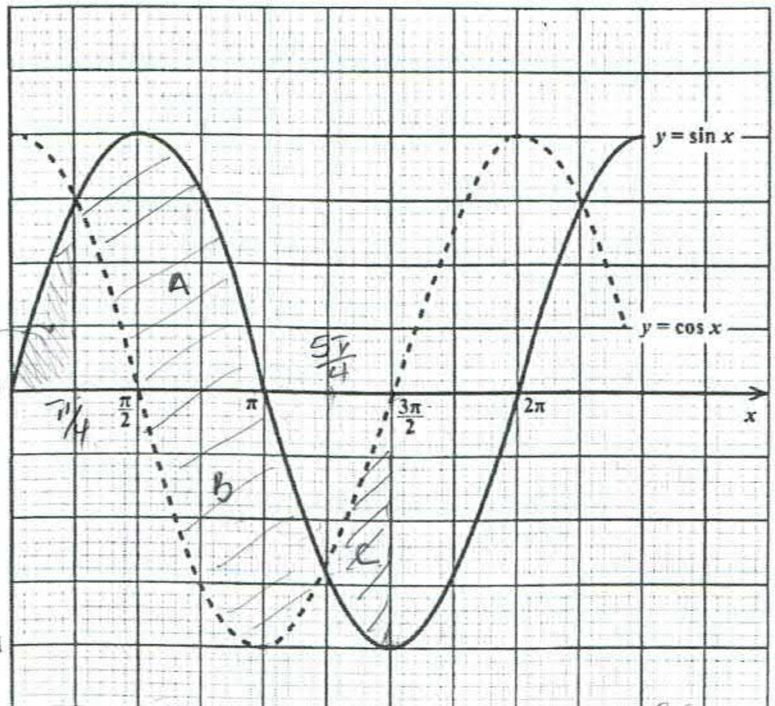
$$= \pi \left[\frac{x^5}{5} + \frac{3x^4}{2} - \frac{4}{3}x^3 + 12x^2 - 32x \right]_0^1$$

$$= \pi \left[\frac{1}{5} + \frac{3}{2} - \frac{4}{3} + 12 - 32 \right] = \frac{589}{30}\pi$$

$$\approx \left| -19\tfrac{19}{30} \right| \pi$$

(b) The diagram below shows the curves $y = \cos x$ and $y = \sin x$. Determine the area bounded by the curves between $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{2}$.

Area $1 - \frac{\sqrt{2}}{2}$



Area of A $= \int_0^{\pi} \sin x - 2\int_0^{\pi/4} \sin x$

$= -\cos x\Big|_0^{\pi}\ 2\left[(-\cos x)_0^{\pi/4}\right]$

$= -(-1)-(-1) - 2\left[-\frac{\sqrt{2}}{2} - (-1)\right]$

$= 2 - 2\left(1 - \frac{\sqrt{2}}{2}\right) = \frac{2\sqrt{2}}{2} = \sqrt{2}$

Area of B = Area of A $= \frac{2\sqrt{2}}{2} = \sqrt{2}$

Area of C $= 1 - 2\left(1 - \frac{\sqrt{2}}{2}\right) = \sqrt{2} - 1$

Total area $= 2\sqrt{2} + (\sqrt{2} - 1) = \underline{3\sqrt{2} - 1}$

GO ON TO THE NEXT PAGE

(c) On the surface of the moon, the acceleration due to gravity is -1.625 m/s^2. A rocket is launched away from the moon with an initial height of 1 200 metres and an initial velocity of 30 m/s.

Determine the velocity and height of the rocket 5 seconds after launch.

$$\frac{dv}{dt} = -1.625$$

$$V = -1.625t + c$$

when $t = 0$ $V = 30$

$$V = -1.625t + 30$$

$$= \frac{dx}{dt} = -1.625t + 30$$

$$x = -\frac{1.625t^2}{2} + 30t + c$$

when $t = 0$ $x = 1200$

$$c = 1200$$

$$x = -\frac{1.625t^2}{2} + 30t + 1200$$

when $t = 5$

$$V = -1.625(5) + 30 = 21.875 \text{ m/s}$$

$$h = 1329.6575 \text{ m}$$

[7 marks]

GO ON TO THE NEXT PAGE

(d) Determine $\int \cos^3 2x \sin 2x \, dx$.

Let $u = \sin 2x$

$\frac{du}{dx} = 2\cos 2x$

$dx = \frac{du}{2\cos 2x}$

$\int \cos^3 2x \frac{u \, du}{2\cos 2x} = \int \cos^2 2x \frac{u}{2} du$

$= \int 1 - \sin^2 2x \frac{u}{2} du$

$= \int (1 - u^2) \frac{u}{2} du$

$= \int \frac{u}{2} - \frac{u^3}{2} du$

$= \frac{u^2}{4} + \frac{u^4}{8} + c = \frac{\sin^2 2x}{4} + \frac{\sin^4 2x}{8} + c$

[4 marks]

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

Probably easier if $u = \cos 2x$!