SECTION A

Module 1

Answer BOTH questions.

1. (a) Two complex numbers are given as $z_1 = \sqrt{\frac{5}{3}} + \sqrt{5}i$ and $z_2 = 2 - 2i$.

(i) Show that the complex number $\frac{z_1}{z_2} = \sqrt{\frac{5}{6}}\, e^{i\frac{7\pi}{12}}$.

$$|z_1| = \sqrt{\frac{5}{3} + 5} = \sqrt{\frac{20}{3}} = 2\sqrt{\frac{5}{3}}$$

$$\arg z_1 = \tan^{-1} \frac{\sqrt{5}}{\sqrt{\frac{5}{3}}} = \tan^{-1}\sqrt{3} = \pi/3$$

$$|z_2| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\arg z_2 = -\tan^{-1} 1 = -\frac{\pi}{4}$$

$$\frac{z_1}{z_2} = \frac{2\sqrt{\frac{5}{3}}\, e^{i\pi/3}}{2\sqrt{2}\, e^{-\pi/4}} = \sqrt{\frac{5}{3}} \times \frac{1}{\sqrt{2}}\, e^{i(\pi/3 + \pi/4)}$$

$$= \sqrt{\frac{5}{6}}\, e^{i\frac{7\pi}{12}}$$

[6 marks]

(ii) **Hence, without using a calculator, determine the value of** $\left(\frac{z_1}{z_2}\right)^2$.

$$\left(\frac{z_1}{z_2}\right)^2 = \left(\sqrt{\frac{5}{6}}\, e^{i\frac{7\pi}{12}}\right)^2$$

$$= \frac{5}{6} e^{i\frac{7\pi}{6}}$$

$$= \frac{5}{6}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$= \frac{5}{6}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= -\frac{5}{12}\left(\sqrt{3} + i\right)$$

[3 marks]

(b) One root of a quadratic equation is given as $4 - 7i$. Determine the quadratic equation with real coefficients which has the root $4 - 7i$.

If $\alpha = 4-7i$

$\beta = 4+7i$

$y = [z-(4-7i)][z-(4+7i)]$

$0 = z^2 - (4-7i)z - z(4+7i) + (4-7i)(4+7i)$

$0 = z^2 - 8z + [16+49]$

$y = z^2 - 8z + 65 = 0$

[4 marks]

(c) Show that the derivative of $\sin^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$ with respect to x is $\dfrac{-1}{\sqrt{(1+\sin x)^2-\cos^2 x}}$.

let $y = \sin^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left[\frac{\cos x}{(1+\sin x)}\right]^2}} \cdot \frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \frac{1}{\sqrt{1-\frac{\cos^2 x}{(1+\sin x)^2}}} \cdot \frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \frac{1}{\sqrt{\frac{(1+\sin x)^2-\cos^2 x}{(1+\sin x)^2}}} \cdot \frac{(1+\sin x)(-\sin x)-\cos x(\cos x)}{(1+\sin x)^2}$$

$$= \frac{1}{\sqrt{\frac{1+2\sin x+\sin^2 x-\cos^2 x}{(1+\sin x)^2}}} \cdot \frac{-1(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{\cancel{(1+\sin x)}}{\sqrt{(1+\sin x)^2-\cos^2 x}} \cdot \frac{-1}{\cancel{1+\sin x}}$$

[6 marks]

$$= \frac{-1}{\sqrt{(1+\sin x)^2-\cos^2 x}}$$

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(d) A curve is defined parametrically by $x = (3 - 2t)^2$, $y = t^3 - 2t$. Determine the equation of the tangent to the curve at the point where $t = 2$.

$x = (3-2t)^2$ $\qquad$ $y = t^3 - 2t$

$\frac{dx}{dt} = 2(3-2t)(-2)$ $\qquad$ $\frac{dy}{dt} = 3t^2 - 2$

$= 8t - 12$

$\frac{dy}{dx} = \frac{3t^2 - 2}{8t - 12}$

when $t = 2$ $\quad x = (3-4)^2 = 1$

$y = 2^3 - 2(2) = 4$

$\frac{dy}{dx} = \frac{3(2^2) - 2}{8(2) - 12} = \frac{10}{4} = \frac{5}{2}$

so equation of tangent

$y - 4 = \frac{5}{2}(x - 1)$

[6 marks]

Total 25 marks

2. (a) Using the substitution $e^x = 3\cos\theta$, or otherwise, determine $\int e^x \sqrt{9 - e^{2x}}\, dx$.

$e^x = 3\cos\theta \qquad e^{2x} = 9\cos^2\theta$

$x = \ln(3\cos\theta)$

$\frac{dx}{d\theta} = \frac{1}{3\cos\theta} \cdot -3\sin\theta$

$\frac{dx}{d\theta} = -\frac{\sin\theta}{\cos\theta}$

so integral becomes

$= \int 3\cos\theta\sqrt{9 - 9\cos^2\theta} \cdot -\frac{\sin\theta}{\cos\theta}\, d\theta$

$= \int -9\sin^2\theta\, d\theta$

$= -9\int \frac{1}{2}(1 - \cos 2\theta)\, d\theta$

$= -\frac{9}{2}\int 1 - \cos 2\theta\, d\theta$

$= -\frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$

$= -\frac{9}{2}\left[\theta - \sin\theta\cos\theta\right] + C$

$= -\frac{9}{2}\left[\cos^{-1}\left(\frac{e^x}{3}\right) - \frac{e^x}{3}\sqrt{1 - \frac{e^{2x}}{9}}\right] + C$

[8 marks]

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(b) (i) Use partial fractions to prove that $\frac{2x+1}{2x^3-x^2+8x-4} = \frac{8}{17(2x-1)} - \frac{4x-15}{17(x^2+4)}$.

$2x^3 - x^2 + 8x - 4$

$x = \frac{1}{2}$ is a root $\Rightarrow (2x-1)$ is a factor

$$\begin{array}{r|l} & x^2 + 4 \\ \hline 2x-1 & 2x^3 - x^2 + 8x - 4 \\ & 2x^3 - x^2 \\ \hline & = \quad = \quad 8x - 4 \\ & 8x - 4 \\ \hline & = \quad = \end{array}$$

$$\frac{2x+1}{(2x-1)(x^2+4)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4) + (Bx+C)(2x-1)}{(2x-1)(x^2+4)}$$

$2 = \frac{17}{4}A \Rightarrow A = \frac{8}{17}$: $x = \frac{1}{2}$

$1 = \frac{32}{17} - C \Rightarrow C = \frac{15}{17}$: $x = 0$

Considering x^2 terms

$A + 2B = 0$

$\frac{8}{17} + 2B = 0 \Rightarrow B = -\frac{4}{17}$

So $\dfrac{2x+1}{2x^3-x^2+8x-4}$

$$= \frac{8}{17(2x-1)} + \frac{\frac{-4}{17}x + \frac{15}{17}}{x^2+4}$$

$$= \frac{8}{17(2x-1)} - \frac{(4x-15)}{17(x^2+4)}$$

[12 marks]

(ii) Hence, find $\int \frac{2x+1}{2x^3-x^2+8x-4}\,dx$.

$$= \int \frac{8}{17(2x+1)} - \frac{4x-15}{17(x^2+4)}\,dx$$

$$= \int \frac{8}{17(2x+1)} - \frac{4x}{17(x^2+4)} + \frac{15}{17(x^2+4)}\,dx$$

$$= \frac{8}{17}\,\frac{\ln|(2x+1)|}{2} - \frac{2}{17}\ln(x^2+4) + \frac{15}{17}\cdot\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right) + c$$

[5 marks]

Total 25 marks

SECTION B

Module 2

Answer BOTH questions.

3. (a) A series is given as $\sum_{k=1}^{n} \frac{8}{4k^2 - 1}$.

(i) Determine the THIRD partial sum, S_3, of the series.

$S_1 = \frac{8}{3}$

$S_2 = \frac{8}{3} + \frac{8}{15}$

$S_3 = \frac{8}{3} + \frac{8}{15} + \frac{8}{35} = \frac{24}{7}$

[3 marks]

(ii) Show that $\sum_{k=1}^{n} \frac{8}{4k^2-1} = \frac{8n}{2n+1}$.

$$\frac{8}{4k^2-1} = \frac{8}{(2k+1)(2k-1)} = 4\left[\frac{1}{2k-1} - \frac{1}{2k+1}\right]$$

$$4\sum \frac{1}{2k-1} - \frac{1}{2k+1}$$

$$= 4\left[\left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{7}}\right) + \cdots \left(\cancel{\tfrac{1}{2n-1}} - \tfrac{1}{2n+1}\right)\right]$$

$$= 4\left[1 - \frac{1}{2n+1}\right]$$

$$= 4\left[\frac{2n+1-1}{2n+1}\right]$$

$$= 4\left[\frac{2n}{2n+1}\right]$$

$$= \frac{8n}{2n+1}$$

[7 marks]

(b) (i) Determine the Taylor series expansion of $e^{\cos x}$ about $x = \frac{\pi}{2}$ up to the term in x^3.

$$f(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$f(x) = e^{\cos x} \quad \therefore f(\pi/2) = 1$$

$$f'(x) = e^{\cos x}(-\sin x) \quad \therefore f'(\pi/2) = -1$$

$$f''(x) = e^{\cos x}(-\cos x) + (-\sin x)\, e^{\cos x}(-\sin x)$$

$$= e^{\cos x}(-\cos x + \sin^2 x) \quad \therefore f''(\pi/2) = 1$$

$$f'''(x) = e^{\cos x}(\sin x + 2\sin x\cos x)$$

$$+ (\sin^2 x - \cos x)\, e^{\cos x}(-\sin x)$$

$$f'''(\pi/2) = 1(1+0) + (1-0)\,1\,(-1)$$

$$= 1 - 1 = 0$$

So

$$e^{\cos x} = 1 - (x - \pi/2) + \frac{1}{2}(x - \pi/2)^2 + \frac{0}{3!}(x - \pi/2)^3 + \dots$$

$$= 1 - \left(x - \frac{\pi}{2}\right) + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$$

[8 marks]

(ii) Use the series expansion to approximate $e^{\cos \pi}$ correct to 2 decimal places.

$$e^{\cos \pi} = 1 - \left(\pi - \frac{\pi}{2}\right) + \frac{1}{2}\left(\pi - \frac{\pi}{2}\right)^2$$

$$= 1 - \frac{\pi}{2} + \frac{\pi^2}{8}$$

$$= 0.6629$$

$$\approx 0.66 \text{ (2 decimal places)}$$

[3 marks]

(c) The first and fifth terms of a geometric progression are 16 and 9, respectively. Determine the THIRD term of the progression.

$a, ar, ar^2 \ldots \ldots ar^{n-1}$

$u_1 = 16$

$u_5 = 16r^4 = 9$

$r^4 = \frac{9}{16} \Rightarrow r^2 = \frac{3}{4}$

$u_3 = 16 \times \frac{3}{4} = 12$

[4 marks]

Total 25 marks

4. (a) Determine the first three terms of the expansion of $(1 - 8x)^{\frac{1}{2}}$.

$$(1-8x)^{1/2}$$

$$= 1 + \frac{1}{2}(-8x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(-8x)^2 + \cdots$$

$$= 1 - 4x - \frac{1}{8}64x^2 - \cdots$$

$$= 1 - 4x - 8x^2$$

[4 marks]

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(b) Hence, by letting $x = \frac{1}{100}$, determine $\sqrt{23}$.

$$\left(1 - \frac{8}{100}\right)^{1/2} = \left(\frac{23}{25}\right)^{1/2} = \frac{\sqrt{23}}{5}$$

$$= 1 - 4\frac{1}{100} - 8\left(\frac{1}{100}\right)^2 \ldots$$

$$= 1 - \frac{1}{25} - \frac{8}{10000} \ldots = \frac{1199}{1250}$$

$$\sqrt{23} = 5 \times \frac{1199}{1250}$$

$$= 4\frac{199}{250} \approx 4.796$$

[6 marks]

(c) (i) Use the Intermediate Value Theorem to show that the equation

$4\sin 2x + x^3 - 3 = 0$ has a root in the interval [0, 1].

let $f(x) = 4\sin 2x + x^3 - 1$

$f(x)$ is continuous on $[0,1]$ since $\sin 2x$ and x^3 are continuous on $[0,1]$

$f(0) = -3$

$f(1) = 3.637 + 1 - 3 = 1.637$

So since $f(0) \times f(1) < 0$ and $f(x)$ is continuous on $[0,1]$ then by the IVT there is is at least one root to $f(x)$ in the interval $[0,1]$

[4 marks]

(ii) Use three iterations of the interval bisection method to obtain an approximation of the root.

$f(x) = 4\sin 2x + x^3 - 3$

$f(0) = -3$

$f(1) = 1.637$

1 mid point $= 0.5$

$f(0.5) = 4\sin 1 + 0.5^3 - 3 = 0.4909$

2 mid point $= 0.25$

$f(0.25) = 4\sin(0.5) + (0.25)^3 - 3 = -1.066$

3 mid point $= 0.375$

$f(0.375) = 4\sin(0.75) + (0.375)^3 - 3 =$

$= -0.2207$

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mid point $= 0.4375$

$f(0.4375) = 0.1539$

mid point $= 0.40625$

$f(0.40625) = -0.0289$

[6 marks]

mid point $= 0.4218$

Approximation of root

$= 0.375$

$\boxed{\text{Root} = 0.411}$

(d) Use the iteration $x_{n+1} = \frac{\sin x_n + 2}{3}$ and the initial approximation $x = 1$ to calculate an approximate value of the root of $f(x) = \sin x - 3x + 2$, correct to 2 decimal places.

$f(x) = \sin x - 3x + 2$

$x_0 = 1$

$x_1 = 0.9472$

$x_2 = 0.93725$

$x_3 = 0.9353$

$x_4 = 0.9349$

$x_5 = 0.93485$

$x_6 = 0.9348$

Root $= 0.93$

[5 marks]

Total 25 marks

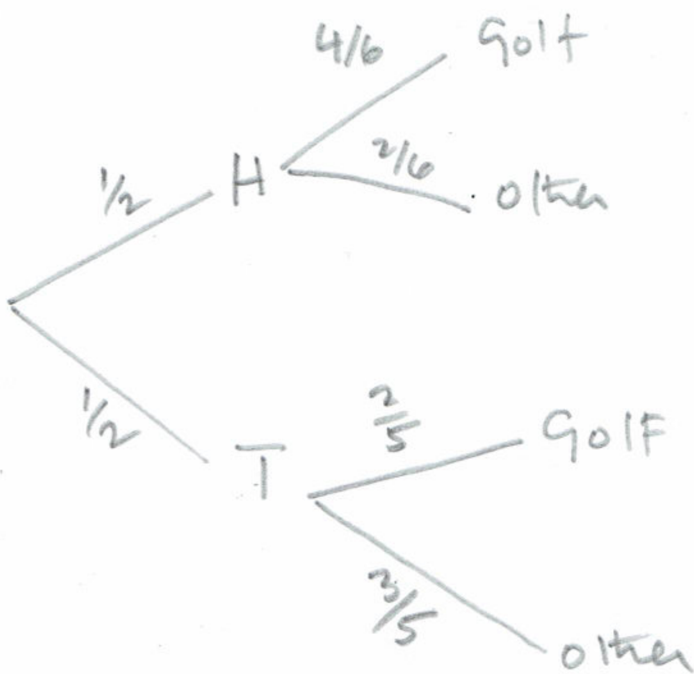
SECTION C

Module 3

Answer BOTH questions.

5. (a) A bag contains 4 golf balls and 2 other balls, whereas a box contains 2 golf balls and 3 other balls. A fair coin is tossed and if the result is a head, a ball is drawn from the bag, otherwise a ball is drawn from the box.

(i) Represent the possible outcomes of a single trial of this experiment on a tree diagram.



[3 marks]

(ii) Determine the probability that a golf ball is drawn on the first trial of the experiment.

$$Pr(\text{Golf ball}) = \left(\frac{1}{2} \times \frac{4}{6}\right) + \left(\frac{1}{2} \times \frac{2}{5}\right)$$

$$= \frac{1}{3} + \frac{1}{5}$$

$$= \frac{8}{15}$$

[4 marks]

(b) (i) A group of twelve persons are to travel in three cars. Each car can seat four persons.

In how many ways can the group be seated in the cars if two particular persons refuse to travel in the same car?

a b c d	e f g h	i j k l
${}^{12}C_4$	${}^{8}C_4$	${}^{4}C_4$
495	70	1

= 34 650 way of seating 12 persons in 3 cars

put the two persons together i.e. ||

$\square$ c d	e f g h	i j k l	= 3150
${}^{10}C_2$	${}^{8}C_4$	1	
c d e f	ab g h	i j k l	= 3150
${}^{10}C_4$	${}^{6}C_2$	1	
c d e f	g h i j	ab k l	= 3150
${}^{10}C_4$	${}^{6}C_2$	1	9450

[5 marks]

So number of ways the 2 are not together

34650 − 9450 = 25,200

(ii) On a table, there is space for 10 books out of a total of 16 available books. However, a Bible and a book of ghost stories must go at the ends.

In how many ways can the books be arranged on the table?

B ______ G

8 spaces

So 14 books to be arranged in 8 spaces

$^{14}P_8$

So no of possible arrangement

$= 2 \times {}^{14}P_8 = 242\ 161\ 920$

[5 marks]

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(c) A system of linear equations with unknowns x, y, z respectively, is represented in the following augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 0 & -3 & -16 \\ -2 & -1 & 3 & 15 \end{array}\right).$$

(i) By reducing the matrix to row echelon form, show that the system has a finite set of solutions.

$$\begin{array}{l} \\ R_2 - 2R_1 \\ 2R_1 + R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & -5 & -22 \\ 0 & 1 & 5 & 21 \end{array}\right]$$

$$\begin{array}{l} \\ \\ 2R_3 + R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & -5 & -22 \\ 0 & 0 & 5 & 20 \end{array}\right]$$

[4 marks]

(ii) Hence, solve the system of linear equations.

$5z = 20$ (From Row 3)

$z = \frac{20}{5} = 4$

$-2y - 20 = -22$ (from Row 2)

$y = 1$

$x + 1 + 4 = 3$ (from Row 1)

$x = -2$

[4 marks]

Total 25 marks

6. (a) A differential equation is given as $\frac{dy}{dx}\tan x + y - 2\cos^2 x = 0$.

(i) Show that the general solution of the differential equation is

$$y = \frac{5}{3} + \frac{1}{3}\cos 2x + C\operatorname{cosec} x, \text{ where } C \text{ is a constant.}$$

$$\frac{dy}{dx} + \frac{1}{\tan x}y = \frac{2\cos^2 x}{\tan x}$$

$$\frac{dy}{dx} + \frac{\cos x}{\sin x}y = \frac{2\cos^3 x}{\sin x}.$$

$$\text{I.F.} = e^{\int \frac{\cos x}{\sin x}dx} = e^{\ln|\sin x|}$$

$$= \sin x.$$

$$\text{so } y\sin x = 2\int \cos^3 x \, dx$$

$$y\sin x = 2\int \cos x(\cos^2 x)\,dx$$

$$= 2\int \cos x(1-\sin^2 x)\,dx$$

$$= 2\int (\cos x - \cos x\sin^2 x)\,dx$$

$$= 2\left(\sin x - \frac{\sin^3 x}{3}\right) + C$$

$$y = 2 - \frac{2}{3}\sin^2 x + \frac{C}{\sin x}$$

$$y = 2 - \frac{2}{3}\sin^2 x + C\operatorname{cosec} x$$

now $\cos 2x = 1 - 2\sin^2 x$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$y = 2 - \frac{2}{3}\left(\frac{1 - \cos 2x}{2}\right) + C\operatorname{cosec} x$$

$$= 2 - \frac{1}{3} + \frac{1}{3}\cos 2x + C\operatorname{cosec} x$$

$$= \frac{5}{3} + \frac{1}{3}\cos 2x + C\operatorname{cosec} x$$

[10 marks]

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(ii) Hence, determine the particular solution given that $y\left(\frac{\pi}{2}\right) = 0$.

Given $y(\pi/2) = 0$

$$0 = \frac{5}{3} + \frac{1}{3}\cos\pi + C\,\mathrm{cosec}\,\pi/2$$

$$0 = \frac{5}{3} + \frac{1}{3}(-1) + C$$

$$0 = \frac{4}{3} + C$$

$$C = -\frac{4}{3}$$

So particular solution

$$y = \frac{5}{3} + \frac{1}{3}\cos 2x - \frac{4}{3}\,\mathrm{cosec}\,x$$

[3 marks]

(b) (i) Determine the general solution of the differential equation $y'' + 2y' + 5y = 0$.

$$y'' + 2y' + 5y = 0$$

$$m^2 + 2m + 5 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2}$$

$$m = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

so $y = e^{-x}(A \sin 2x + B \cos 2x)$

[7 marks]

(ii) Hence, determine the solution of the boundary value problem $y'' + 2y' + 5y = 0$ with $y(0) = 1$, $y(\pi) = 2$.

Given $y(0) = 1$

$1 = e^{-0}(A \sin 0 + B \cos 0)$

$B = 1$

Given $y(\pi) = 2$

$2 = e^{-\pi}(A \sin 2\pi + \cos 2\pi)$

$2 = e^{-\pi}(A(0) + 1)$

$2 = e^{-\pi}$?!!

[5 marks]

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.