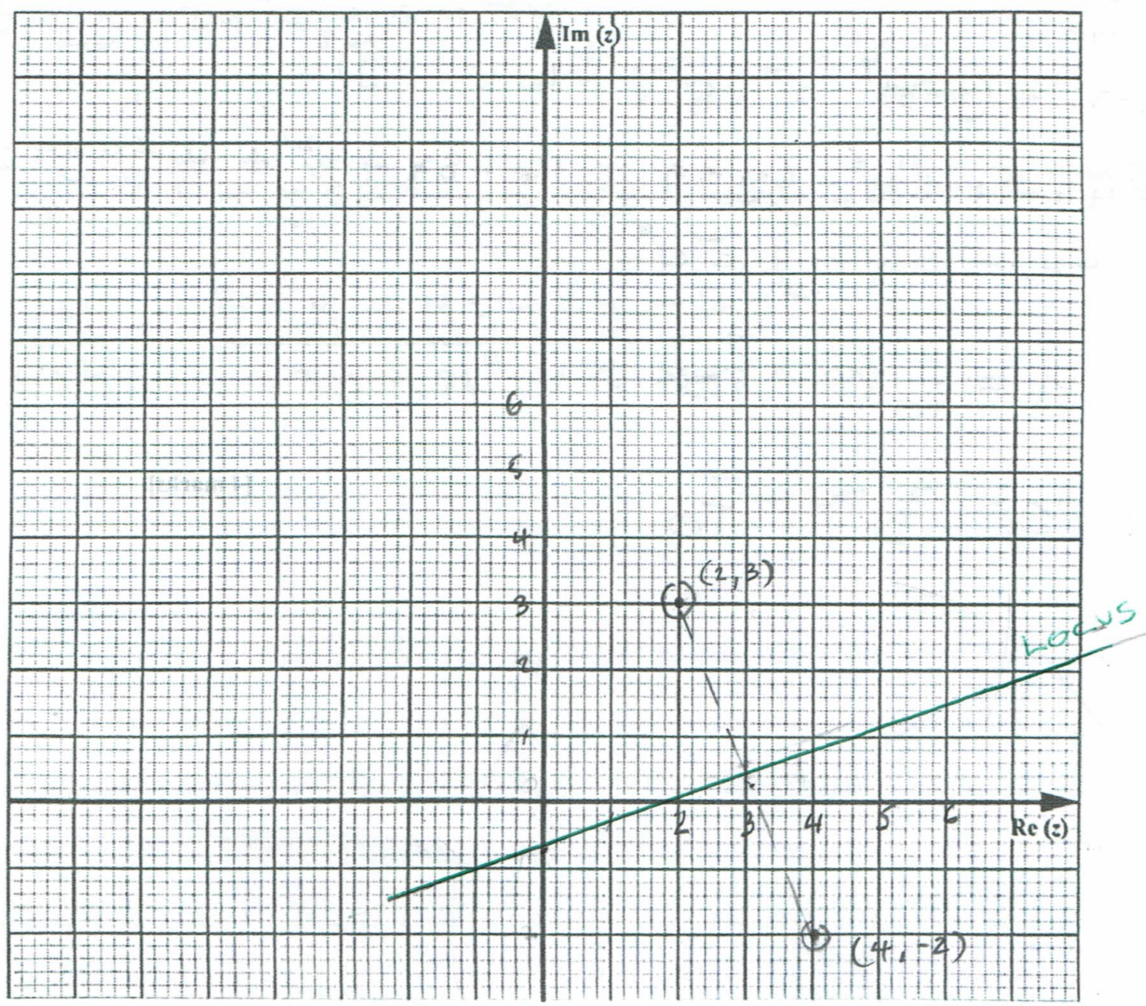
SECTION A

Module 1

Answer BOTH questions.

1. (a) A point z moves in the complex plane such that $|z-2-3i| = |4-2i-z|$.

(i) Sketch the locus of z on the Argand diagram below. **[3 marks]**



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(ii) Determine the Cartesian equation of the locus of z.

let $z = x + iy$

$|z - 2 - 3i| = |4 - 2i - z|$

$|(x+iy) - 2 - 3i| = |4 - 2i - (x+iy)|$

$|(x-2) + (y-3)i| = |(4-x) + (-2-y)i|$

$\therefore (x-2)^2 + (y-3)^3 = (4-x)^2 + (-2-y)^2$

$\cancel{x^2} - 4x + 4 + \cancel{y^2} - 6y + 9 = 16 - 8x + \cancel{x^2} + 4 + 4y + \cancel{y^2}$

$4x - 10y - 7 = 0$

$y = \frac{4}{10}x - \frac{7}{10}$

[4 marks]

$y = \frac{2}{5}x - \frac{7}{10}$

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(b) Given that $x^3y - 2xy^2 = \cos(xy)$, determine $\frac{dy}{dx}$.

differentiating implicitly

$$3x^2y + x^3\frac{dy}{dx} - 2y^2 - 2x(2y)\frac{dy}{dx} = -\sin(xy)\left[y + x\frac{dy}{dx}\right]$$

$$3x^2y - 2y^2 + y\sin xy = 4xy\frac{dy}{dx} - x^3\frac{dy}{dx} - x\sin(xy)\frac{dy}{dx}$$

$$3x^2y - 2y^2 + y\sin xy = \frac{dy}{dx}\left[4xy - x^3 - x\sin(xy)\right]$$

$$\frac{dy}{dx} = \frac{3x^2y + y\sin xy - 2y^2}{4xy - x^3 - x\sin(xy)}$$

[5 marks]

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(c) Given that $3z^2 - pz + 2q = 0$ has root $3 - 5i$, determine the values of p and q where $p, q \in R$.

Given one root is $3-5i$ then the other root is $3+5i$

$\alpha = 3-5i \quad \beta = 3-5i$

sum of roots $= \frac{p}{3} = 3+5i+3-5i = 6$

$\boxed{p = 3 \times 6 = 18}$

product of roots $= \frac{2q}{3} = (3+5i)(3-5i)$

$\frac{2q}{3} = 9 - 25i^2 = 9 + 25 = 34$

$\boxed{q = \frac{34 \times 3}{2} = 51}$

[7 marks]

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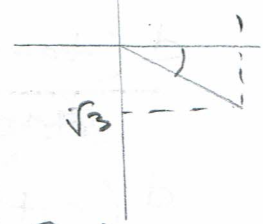
(d) Using DeMoivre's theorem, determine the square root of $1 - i\sqrt{3}$, expressing your answer in radians.

Let $z = 1 - i\sqrt{3}$

$|z| = \sqrt{1 + (\sqrt{3})^2} = \sqrt{4} = 2$



$\tan \alpha = \sqrt{3}$

$\arg z = -\pi/3$

So $z = 2(\cos(^-\pi/3) + i \sin(^-\pi/3)$

$= 2 \cos(\pi/3) - i \sin(\pi/3))$

$z^{1/2} = [2 \cos(\pi/3) - i \sin(\pi/3))^{1/2}$

Using DeMoivre theorem

$z^{1/2} = 2^{1/2}\left[\cos\left(\frac{\pi}{3} \times \frac{1}{2}\right) - i \sin\left(\frac{\pi}{3} \times \frac{1}{2}\right)\right]$

$= \sqrt{2}\,[\cos \pi/6 - i \sin \pi/6]$

$= \sqrt{2}\left[\frac{\sqrt{3}}{2} - \frac{1}{2} i\right]$

$= \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}} i$ //

[6 marks]

Total 25 marks

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2. (a) Use integration by parts to evaluate $\int \sec^2 x \operatorname{cosec}^2 x \, dx$.

$$\int uv \, dx = u\int v - \int \int v \frac{du}{dx} \, dx$$

$$\frac{d(\tan x)}{dx} = \sec^2 x \Rightarrow \int \sec^2 x = \tan x$$

$$d(\cot x) = -\operatorname{cosec}^2 x \Rightarrow \int \operatorname{cosec}^2 x = -\cot x$$

let $u = \sec^2 x$ and $v = \operatorname{cosec}^2 x$

so $\int \sec^2 x \operatorname{cosec}^2 x \, dx$

$$= \sec^2 x \int \operatorname{cosec}^2 x \, dx - \int \int \operatorname{cosec}^2 x \frac{d(\sec^2 x)}{dx} \, dx$$

$$= -\sec^2 x \cot x - \int -\cot x \frac{d(\sec^2 x)}{dx} \, dx \quad (1)$$

now $y = \sec^2 x = (\cos x)^{-2}$

$$\frac{dy}{dx} = -2(\cos x)^{-3}(-\sin x)$$

$$= 2\frac{\sin x}{\cos^3 x}$$

so (1) becomes

$$-\sec^2 x \cot x + 2\int \cot x \cdot \frac{\sin x}{\cos^3 x} \, dx$$

$$= -\sec^2 x \cot x + 2\int \frac{1}{\cos^2 x} \, dx$$

[6 marks]

$$= -\sec^2 x \cot x + 2\tan x + c$$

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$$= -\frac{1}{\cos x \sin x} + 2\frac{\sin x}{\cos x} + c$$

$$\frac{-1 + 2\sin^2 x}{\cos x \sin x} + c = \frac{-\cos(2x)}{\frac{1}{2}\sin(2x)} + c$$

$$= -2\cot(2x) + c$$

(b) Evaluate $\int_2^5 \frac{3x}{x^2 - 4x + 5}\, dx$.

$$x^2 - 4x + 5 = (x-2)^2 + 1$$

$$\text{So } \int_2^5 \frac{3x}{(x-2)^2+1}\, dx$$

$$\text{let } u = x - 2 \implies x = u + 2 \quad \frac{dx}{du} = 1$$

So integral becomes

$$\int_0^3 \frac{3(u+2)}{u^2+1}\, du = \int_0^3 \frac{3u}{u^2+1}\, du + 6\int \frac{1}{u^2+1}\, du$$

$$\int_0^3 \frac{3u}{u^2+1}\, du = \frac{3}{2}\ln(u^2+1)\Big|_0^3 = \frac{3}{2}\ln 10$$

$$\int_0^3 \frac{1}{u^2+1}\, du = \tan^{-1} u \Big|_0^3 = \tan^{-1} 3$$

$$\text{So } \int_2^5 \frac{3x}{x^2 - 4x + 5}\, dx$$

$$= \frac{3}{2}\ln 10 + 6\tan^{-1} 3$$

[9 marks]

(c) Determine $\int \frac{x+1}{x^3-9x}\,dx$. $= \int \frac{x+1}{x(x^2-9)}\,dx$

$= \int \frac{x+1}{x(x+3)(x-3)}\,dx$

$= \int \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-3}\,dx$

$A(x+3)(x-3) + B(x)(x-3) + C(x)(x+3) = x+1$

$x=3$: $18C = 4 \Rightarrow C = \frac{4}{18} = 2/9$

$x=0$: $-9A = 1 \Rightarrow A = -1/9$

$x=-3$: $18B = -2 \Rightarrow B = -\frac{2}{18} = -\frac{1}{9}$

so integral becomes

$\int -\frac{1}{9x} - \frac{1}{9(x+3)} + \frac{2}{9(x-3)}\,dx$

$= -\frac{1}{9}\ln x - \frac{1}{9}\ln(x+3) + \frac{2}{9}\ln(x-3) + C$

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SECTION B

Module 2

Answer BOTH questions.

3. (a) Determine the coefficient of x^4 in the binomial expansion of $\frac{1-2x}{(1+3x)^2}$.

$(1-2x)(1+3x)^{-2}$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

$(1+3x)^{-2} = 1 - 2(3x) + \frac{(-2)(-3)}{2!}(3x)^2 + \frac{(-2)(-3)(-4)}{3!}(3x)^3 + \dots$

$= 1 - 6x + 27x^2 - 108x^3 + 405x^4$

$(1-2x)(1+3x)^{-2} = (1-2x)(1-6x+27x^2-108x^3+405x^4)$

coefficient of x^4

$405x^4 + 216x^4 = 621x^4$

coefficient of x^4 = 621

[7 marks]

(b) Show that if the series $1 + \left(\frac{7}{3x-5}\right) + \left(\frac{7}{3x-5}\right)^2 + \left(\frac{7}{3x-5}\right)^3 + \ldots$ converges, then its sum to infinity is $\frac{3x-5}{3x-12}$.

$a = 1 \qquad r = \frac{7}{3x-5}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1 - \frac{7}{3x-5}}$

$= \frac{1}{\frac{3x-5-7}{3x-5}} = \frac{1}{\frac{3x-12}{3x-5}}$

$= \frac{3x-5}{3x-12}$

[8 marks] ??

(c) Use mathematical induction to prove that $\frac{d^n}{dx^n} e^x \sin x = 2^{\frac{n}{2}} e^x \sin\left(x + \frac{n\pi}{4}\right)$.

You may use the fact that $\sin x + \cos x = \sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right)$ and that

$$\frac{d^{k+1}}{dx^{k+1}} f(x) = \frac{d}{dx}\left[\frac{d^k}{dx^k} f(x)\right].$$

Let P(n) be the statement that

$$\frac{d^n}{dx^n} e^x \sin x = 2^{n/2} e^x \sin\left(x + \frac{n\pi}{4}\right)$$

$$P_1 \quad \frac{d}{dx} e^x \sin x = e^x \cos x + e^x \sin x$$

$$= e^x (\cos x + \sin x)$$

$$= e^x (\sqrt{2} (\sin (x + \frac{\pi}{4})$$

$$= 2^{1/2} \sin(x + \frac{\pi}{4})$$

So P(1) is true.

Assume P_k is true i.e.

$$\frac{d^k}{dx^k} e^x \sin x = 2^{k/2} e^x \sin\left(x + \frac{k\pi}{4}\right)$$

required to show that

$$\frac{d^{k+1}}{dx^{k+1}} e^x \sin x = 2^{k+1/2} e^x \sin\left(x + \frac{(k+1)\pi}{4}\right)$$

now $\frac{d^{k+1}}{dx^{k+1}} e^x \sin x = \frac{d}{dx}\left(\frac{d^k}{dx^k} e^x \sin x\right)$

$= \frac{d}{dx}\left[2^{k/2} e^x \sin\left(x + \frac{k\pi}{4}\right)\right]$

$= 2^{k/2}\left[e^x \cos\left(x+\frac{k\pi}{4}\right) + e^x \sin\left(x+\frac{k\pi}{4}\right)\right]$

$= 2^{k/2} e^x \left[\cos\left(x+\frac{k\pi}{4}\right) + \sin\left(x+\frac{k\pi}{4}\right)\right]$ (1)

now $\sin A + \cos A = \sqrt{2}\sin\left(A + \frac{\pi}{4}\right)$

So $\cos\left(x+\frac{k\pi}{4}\right) + \sin\left(x+\frac{k\pi}{4}\right)$

$= \sqrt{2}\sin\left(x + \frac{k\pi}{4} + \frac{\pi}{4}\right) = \sqrt{2}\sin\left(x + \frac{k+1}{4}\pi\right)$

So (1)

$= 2^{k/2} e^x 2^{1/2} \sin\left(x + \left(\frac{k+1}{4}\right)\pi\right)$

$= 2^{\frac{k+1}{2}} e^x \sin\left(x + \frac{k+1}{4}\pi\right)$

So P_{k+1} is true if P_k is true.

Conclusion: since P_1 is true and P_{k+1} is true when P_k is true then by M.I. P_n is true for all $n \in \mathbb{N}$.

[10 marks]

Total 25 marks

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4. (a) (i) Use the intermediate value theorem to prove that $x^5 - 2x^3 + x^2 + 2 = 0$ has a root in the interval $[-2, -1.5]$.

Let $f(x) = x^5 - 2x^3 + x^2 + 2$

$f(x)$ is continuous

$f(-2) = -32 + 16 + 4 + 2 = -10 < 0$

$f(-1.5) = -7.59375 + 6.75 + 2.25 + 2$

$= 3.40625 > 0$

so by the I.V.T $\exists$ root in $[-2, -1.5]$

[3 marks]

(ii) Hence, use two iterations of the Newton–Raphson method, with initial estimate $x_1 = -1.5$, to calculate a new estimate of the root of $x^5 - 2x^3 + x^2 + 2 = 0$ in the interval $[-2, -1.5]$.

$f(x) = x^5 - 2x^3 + x^2 + 2$ $\qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$f'(x) = 5x^4 - 6x^2 + 2x$

$x_1 = -1.5$

$x_2 = -1.5 - \frac{f(-1.5)}{f'(-1.5)} = -1.5 - \frac{3.40625}{(25.3125 - 13.5 - 3)} = -1.5 - \frac{3.40625}{8.8125}$

$= -1.886524823$

$x_3 = -1.886524823 -$

[6 marks]

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(b) (i) Show that $\theta_{n+1} = \sin^{-1}\left(\frac{2}{8-\theta_n^2}\right)$ is a suitable iteration for the approximation of the roots of the equation $\operatorname{cosec}\theta = 4 - \frac{1}{2}\theta^2$.

$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = 4 - \frac{1}{2}\theta^2 = \frac{8-\theta^2}{2}$

$= \sin\theta = \frac{2}{8-\theta^2}$

$\theta = \sin^{-1}\left(\frac{2}{8-\theta^2}\right)$

[4 marks]

(ii) Hence, using $\theta_1 = 0$ as the first approximation, calculate a new estimate for the root of $\operatorname{cosec}\theta = 4 - \frac{1}{2}\theta^2$, correct to three decimal places.

$\theta_1 = 0$ $\quad\theta_2 = 0.252680255$

$\theta_3 = 0.254758055$

$\theta_4 = 0.254792657$

root $= \underline{0.255}$

[5 marks]

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(c) Determine the Maclaurin expansion of $f(x) = (1 + x^2)\cos x$ up to and including the third non-zero term.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 +$$

$$f(x) = (1+x^2)\cos x \qquad f(0) = 1$$

$$f'(x) = -(1+x^2)\sin x + 2x\cos x \qquad f'(0) = 0$$

$$f''(x) = -(1+x^2)\cos x + \sin x(-2x) - 2x\sin x + 2\cos x$$

$$= -(1+x^2)\cos x - 4x\sin x + 2\cos x$$

$$= \cos x - x^2\cos x - 4x\sin x \qquad f''(0) = 1$$

$$f'''(x) = -\sin x + \sin x\, x^2 - 2x\cos x - 4x\cos x - 4\sin x$$

$$f'''(0) = 0$$

$$f'''(x) = -5\sin x + x^2\sin x - 6x\cos x$$

$$f^{IV}(x) = -5\cos x + x^2\cos x + 2x\sin x + 6x\sin x - 6\cos x$$

$$f^{IV}(0) = -5 - 6 = -11$$

$$f(x) = 1 + \frac{1}{2}x^2 - \frac{11}{24}x^4$$

[7 marks]

Total 25 marks

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SECTION C

Module 3

Answer BOTH questions.

5. (a) A matrix is given as $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -k \\ 1 & -k & -1 \end{bmatrix}$.

(i) Calculate the determinant of M.

$$|M| = 1\begin{vmatrix} 2 & -k \\ -k & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -k \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & -k \end{vmatrix}$$

$$= (-2-k^2) - 1(-1+k) - 1(-k-2)$$

$$= \cancel{-2} - k^2 + 1 \cancel{-k} \cancel{+k} \cancel{+2}$$

$$= 1 - k^2$$

[5 marks]

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(ii) Hence, or otherwise, determine the values of k for which the simultaneous equations

$$x + y - z = 1$$
$$x + 2y - kz = 0$$
$$x - ky - z = 1$$

have a unique solution.

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -k \\ 1 & -k & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

for a unique solution $|M| \neq 0$

$1 - k^2 \neq 0$

$k^2 \neq 1$

$k \neq \pm 1$

[4 marks]

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(iii) Using $k = 2$ in the matrix, M, solve the system of linear equations by first reducing it to row echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & 2 & -2 & 0 \\ 1 & -2 & -1 & 1 \end{array}\right)$$

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -3 & 0 & 0 \end{array}\right)$$

$$\begin{array}{l} \\ \\ R_3 + 3R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -3 & -3 \end{array}\right)$$

$-3z = -3 \implies z = +1$

$y - z = -1$

$y - 1 = -1 \implies y = 0$

$x + y - z = 1$

$x + 0 - 1 = 1 \implies x = 2$

[6 marks]

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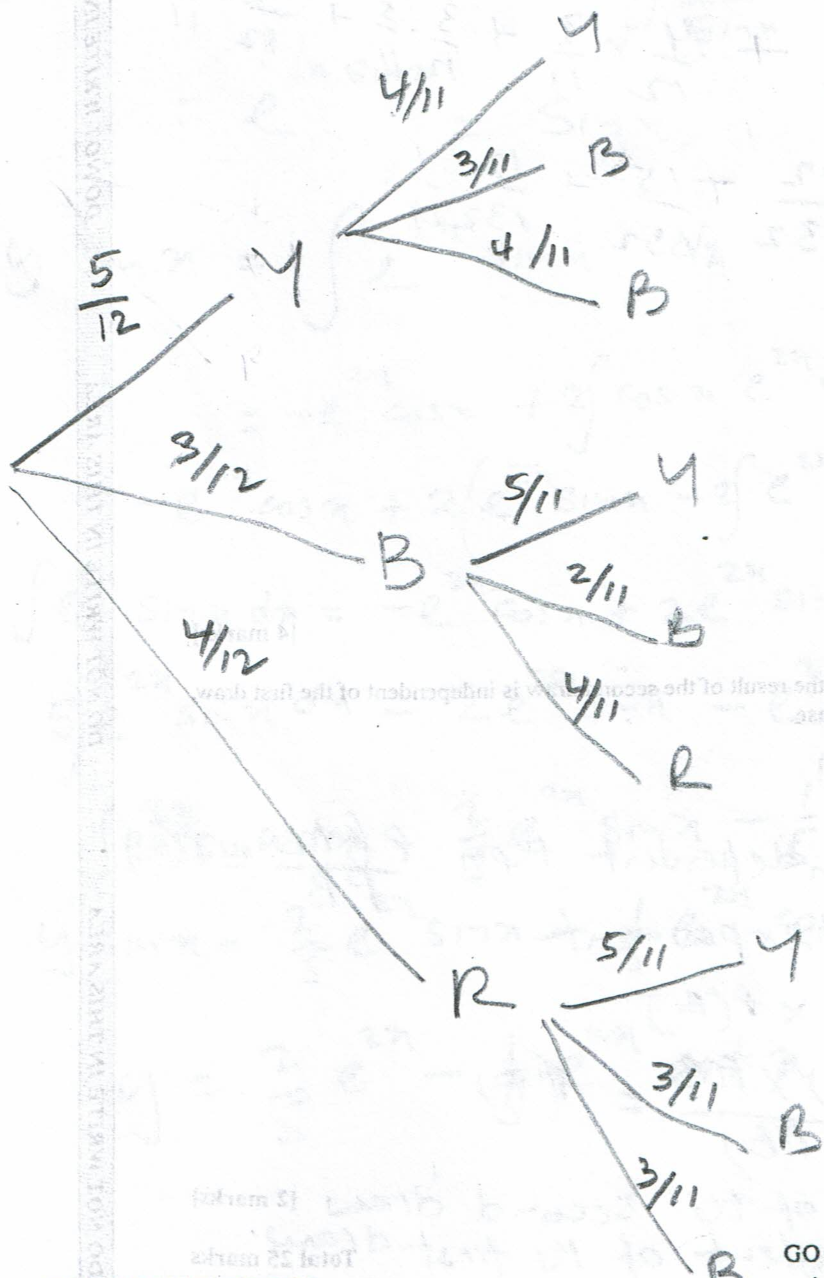
(b) A bag contains five yellow pencils, three blue pencils and four red pencils. Two pencils are taken out of the bag, at random, and are not replaced.

(i) Draw a tree diagram to represent the possible events and their respective probabilities.



[4 marks]

(ii) Determine the probability that at least one pencil taken from the bag is blue.

$P(YB) + P(BB) + P(RB) + P(BY) + P(BR)$

$= \left(\frac{5}{12} \times \frac{3}{11}\right) + \frac{3}{12} \times \frac{2}{11} + \frac{4}{12} \times \frac{3}{11} + \frac{3}{12} \cdot \frac{5}{11} + \frac{3}{12} \cdot \frac{4}{11}$

$= \frac{15}{132} + \frac{6}{132} + \frac{12}{132} + \frac{15}{132} + \frac{12}{132}$

$= \frac{60}{132} = \frac{5}{11}$

[4 marks]

(iii) Determine whether the result of the second draw is independent of the first draw. **Justify your response.**

$P(A/B) = \frac{P(A \cap B)}{P(B)}$

If A and B are independent then $\frac{P(A \cap B)}{PB} = P(A)$

now for the given experiment

$P(A \cap B) = P(A) \times P(B)$

so $P(A/B) = \frac{P(A) \times P(B)}{P(B)} = P(A)$

so the result of the second draw $(A \cap B)$ is independent of the first draw.

[2 marks]

Total 25 marks

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(ii) No, they are not independent

In each case, the probability of the second outcome changes depending on what was selected first

6. (a) Determine the general solution of the differential equation $y' + y \cot x = e^{2x}$.

$$\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} = e^{2x}$$

$$I.F = e^{\int \cot x \, dx} \qquad \int \frac{\cos x}{\sin x} dx = \ln \sin x$$

$$= e^{\ln \sin x} = \sin x$$

$$y \sin x = \int e^{2x} \sin x \, dx$$

$$= -e^{2x}\cos x + 2\int \cos x \, e^{2x} dx$$

$$= -e^{2x}\cos x + 2\left(e^{2x}\sin x - 2\int e^{2x}\sin x \, dx\right.$$

$$\int e^{2x}\sin x \, dx = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x \, dx$$

$$5\int e^{2x}\sin x \, dx = 2e^{2x}\sin x - e^{2x}\cos x + C$$

$$\int e^{2x}\sin x \, dx = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + C$$

$$y \sin x = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + C$$

$$y = \frac{2}{5}e^{2x} - \frac{1}{5}e^{2x}\cot x + \frac{C}{\sin x}$$

[9 marks]

(b) (i) Show that the general solution of the differential equation $y'' - y' - 2y = 3e^{2x}$ is $y = Ae^{2x} + Be^{-x} + xe^{2x}$.

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0 \Rightarrow m = 2 \quad m = -1$

$y = Ae^{2x} + Be^{-x} \quad (C.F)$

for particular integral try

$y = cxe^{2x}$

$\frac{dy}{dx} = cx(2e^{2x}) + ce^{2x}$

$= 2cxe^{2x} + ce^{2x}$

$\frac{d^2y}{dx^2} = 2cx(2e^{2x}) + 2ce^{2x} + 2ce^{2x}$

$= 4cxe^{2x} + 4ce^{2x}$

$4cxe^{2x} + 4ce^{2x} - (2cxe^{2x} + ce^{2x}) - 2cxe^{x} = 3e^{2x}$

$4c - c = 3$

$3c = 3$

$c = 1$

$y = Ae^{2x} + Be^{-x} + xe^{2x}$

[11 marks]

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(ii) Hence, solve the differential equation given that at $x = 0$, $y = 0$ and $y' = 7$.

$x = 0 \quad y = 0$

$A + B = 0$

$x = 0 \quad \frac{dy}{dx} = 7$

$\frac{dy}{dx} = 2Ae^{x} - Be^{-x} + 2xe^{2x} + e^{2x}$

$2A - B + 1 = 7$

$2A - B = 6$

Solving simultaneously

$A + B = 0$

$2A - B = 6$

$3A = 6 \Rightarrow A = 2$

$B = -2$

$y = 2e^{2x} - 2e^{-x} + xe^{2x}$

[5 marks]

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.