CARIBBEAN EXAMINATIONS COUNCIL

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 2 – Paper 02

ANALYSIS, MATRICES AND COMPLEX NUMBERS

2 hours 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.

2. Each section consists of TWO questions.

3. Answer ALL questions from the THREE sections.

4. Write your answers in the spaces provided in this booklet.

5. Do NOT write in the margins.

6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.

7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**

8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

SECTION A

Module 1

Answer BOTH questions.

1. (a) Two complex numbers are given as $z_1 = \sqrt{\dfrac{5}{3}} + \sqrt{5}i$ and $z_2 = 2 - 2i$.

    (i) Show that the complex number $\dfrac{z_1}{z_2} = \sqrt{\dfrac{5}{6}}\, e^{i\frac{7\pi}{12}}$.

    [6 marks]

(ii)   Hence, **without using a calculator, determine the value of** $\left(\dfrac{z_1}{z_2}\right)^2$.

[3 marks]

(b)   One root of a quadratic equation is given as $4 - 7i$. Determine the quadratic equation with real coefficients which has the root $4 - 7i$.

[4 marks]

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(c)   Show that the derivative of $\sin^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$ with respect to x is $\dfrac{-1}{\sqrt{(1+\sin x)^2 - \cos^2 x}}$.

[6 marks]

(d) A curve is defined parametrically by $x = (3 - 2t)^2$, $y = t^3 - 2t$. Determine the equation of the tangent to the curve at the point where $t = 2$.

[6 marks]

Total 25 marks

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0 2 2 3 4 0 2 0 0 8

2.   (a)   Using the substitution $e^x = 3 \cos \theta$, or otherwise, determine $\int e^x \sqrt{9 - e^{2x}} \; dx$.

[8 marks]

(b)   (i)   Use partial fractions to prove that $\dfrac{2x+1}{2x^3-x^2+8x-4} \equiv \dfrac{8}{17(2x-1)} - \dfrac{4x-15}{17(x^2+4)}$.

[12 marks]

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(ii)   Hence, find $\int \dfrac{2x+1}{2x^3 - x^2 + 8x - 4}\, dx$.

[5 marks]

Total 25 marks

SECTION B

Module 2

Answer BOTH questions.

3. (a) A series is given as $\displaystyle\sum_{k=1}^{n} \frac{8}{4k^2 - 1}$.

    (i) Determine the THIRD partial sum, S_3, of the series.

[3 marks]

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(ii)   Show that $\displaystyle\sum_{k=1}^{n} \frac{8}{4k^2-1} = \frac{8n}{2n+1}$.

[7 marks]

(b) (i) Determine the Taylor series expansion of $e^{\cos x}$ about $x = \dfrac{\pi}{2}$ up to the term in x^3.

[8 marks]

(ii) Use the series expansion to approximate $e^{\cos x}$ correct to 2 decimal places.

[3 marks]

(c)   The first and fifth terms of a geometric progression are 16 and 9, respectively. Determine the THIRD term of the progression.

[4 marks]

Total 25 marks

4.   **(a)**   Determine the first three terms of the expansion of $(1 - 8x)^{\frac{1}{2}}$.

[4 marks]

(b)   Hence, by letting $x = \dfrac{1}{100}$, determine $\sqrt{23}$.

[6 marks]

(c) (i) Use the Intermediate Value Theorem to show that the equation

$4 \sin 2x + x^3 - 3 = 0$ has a root in the interval [0, 1].

[4 marks]

(ii)   Use three iterations of the interval bisection method to obtain an approximation of the root.

[6 marks]

(d)   Use the iteration $x_{n+1} = \dfrac{\sin x_n + 2}{3}$ and the initial approximation $x = 1$ to calculate an approximate value of the root of $f(x) = \sin x - 3x + 2$, correct to 2 decimal places.

[5 marks]

Total 25 marks

SECTION C

Module 3

Answer BOTH questions.

5. (a) A bag contains 4 golf balls and 2 other balls, whereas a box contains 2 golf balls and 3 other balls. A fair coin is tossed and if the result is a head, a ball is drawn from the bag, otherwise a ball is drawn from the box.

   (i) Represent the possible outcomes of a single trial of this experiment on a tree diagram.

[3 marks]

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(ii)   Determine the probability that a golf ball is drawn on the first trial of the experiment.

[4 marks]

(b)   (i)   A group of twelve persons are to travel in three cars. Each car can seat four persons.

In how many ways can the group be seated in the cars if two particular persons refuse to travel in the same car?

[5 marks]

(ii)   On a table, there is space for 10 books out of a total of 16 available books. However, a Bible and a book of ghost stories must go at the ends.

In how many ways can the books be arranged on the table?

[5 marks]

(c) A system of linear equations with unknowns x, y, z respectively, is represented in the following augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 0 & -3 & -16 \\ -2 & -1 & 3 & 15 \end{array} \right).$$

    (i) By reducing the matrix to row echelon form, show that the system has a finite set of solutions.

[4 marks]

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(ii)   Hence, solve the system of linear equations.

[4 marks]

Total 25 marks

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6.   (a)   A differential equation is given as $\dfrac{dy}{dx} \tan x + y - 2 \cos^2 x = 0$.

             (i)   Show that the general solution of the differential equation is

$$y = \frac{5}{3} + \frac{1}{3} \cos 2x + C \operatorname{cosec} x, \text{ where } C \text{ is a constant.}$$

[10 marks]

(ii) Hence, determine the particular solution given that $y\left(\dfrac{\pi}{2}\right) = 0$.

[3 marks]

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(b)   (i)   Determine the general solution of the differential equation $y'' + 2y' + 5y = 0$.

[7 marks]

(ii)   Hence, determine the solution of the boundary value problem $y'' + 2y' + 5y = 0$ with $y(0) = 1$, $y(\pi) = 2$.

[5 marks]

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.