TEST CODE **02234032**

FORM TP 2020331

MAY/JUNE 2020

C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 2 – Paper 032

ANALYSIS, MATRICES AND COMPLEX NUMBERS

1 hour 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Each section consists of ONE question.
3. Answer ALL questions.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

“*”Barcode Area”*”
Sequential Bar Code

SECTION A

Module 1

Answer this question.

1. (a) A rectangular box has dimensions, x, y, z each of which is measured with a maximum error of 0.1 cm.

Determine the maximum error of the volume, V, of the box given that $x = 75$ cm, $y = 60$ cm, $z = 40$ cm and the maximum error is $dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz$.

Volume $= xyz$

$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = dv$

$dv = yz(0.1) + xz(0.1) + xy(0.1)$

$= (60 \times 40)(0.1) + (75 \times 40)\,0.1 + (75 \times 60)(0.1)$

$= 990 \text{ cm}^3$

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(b) Determine z such that $z^2 - (6 + 2i)\,z + (7 + 6i) = 0$.

$$z = \frac{6+2i \pm \sqrt{(6+2i)^2 - 4(7+6i)}}{2}$$

$$z = \frac{6+2i \pm \sqrt{36+24i-4-28-24i}}{2}$$

$$= \frac{6+2i \pm \sqrt{4}}{2}$$

$$= \frac{6+2i \pm 2}{2}$$

$z = 4+i$ and $z = 2+i$

[6 marks]

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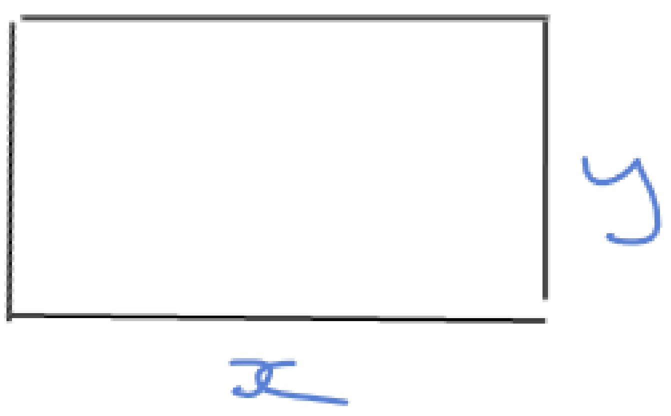
(c) Given rectangles with a fixed perimeter, p, show that a square has the **maximum** area.



$$P = 2(x+y)$$
$$P = 2x + 2y$$
$$2y = P - 2x$$
$$y = \frac{P-2x}{2} = \frac{P}{2} - x$$

$$\text{Area} = xy = x\left(\frac{P}{2} - x\right) = \frac{Px}{2} - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x \Rightarrow x = \frac{P}{4}$$

$$\frac{d^2A}{dx^2} = -2$$

when $x = \frac{P}{4}$ $\quad y = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$

So $x = y$ and since $\frac{d^2A}{dx^2} = -2 \Rightarrow$ max.

[8 marks]

Total 20 marks

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SECTION B

Module 2

Answer this question.

2. (a) Use mathematical induction to prove that $\frac{1}{2}+\frac{1}{6}+\ldots+\frac{1}{n(n-1)}=1-\frac{1}{n}, n \geq 2$.

Let P(n) be the statement that

$\frac{1}{2}+\frac{1}{6}+\cdots+\frac{1}{n(n-1)}=1-\frac{1}{n}, n \geqslant 2$

for $n=2$: LHS. $\frac{1}{2(2-1)}=\frac{1}{2}$

RHS $1-\frac{1}{2}=\frac{1}{2}$

So $P(2)$ is true.

Assume $k=n$ i.e. $P_k=\sum_{2}^{k} \frac{1}{r(r-1)}=1-\frac{1}{k}$

then $P(k+1)=\sum_{2}^{k} \frac{1}{r(r-1)}+\frac{1}{(k+1)(k)}$

$=1-\frac{1}{k}+\frac{1}{(k+1)k}$

$=\frac{k(k+1)-(k+1)+1}{k(k+1)}=\frac{k^2+k-k-1+1}{k(k+1)}$

$=\frac{k^2}{k(k+1)}=\frac{k}{k+1}$

$$\frac{k}{k+1} = 1 - \frac{1}{k+1}$$

So P(k+1) is true when P(k) is assumed to be true

Hence since P(2) is true and P(k+1) is true when P(k) is assumed true then P(n) is true for all $n \geqslant 2$

[9 marks]

(b) The function $f(x) = 5\ln x - x + 2$ has a root in the interval [0, 1]. By letting $x_1 = 1$ and using the Newton-Raphson method, show that an approximation of the root of f in the interval [0, 1] can be expressed as $\frac{3}{4} - \frac{15}{17}\left(\frac{1}{4} + \ln\frac{3}{4}\right)$.

$$f(x) = 5\ln x - x + 2$$
$$f'(x) = \frac{5}{x} - 1$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = 1$$
$$x_2 = 1 - \frac{5\ln 1 - 1 + 2}{\frac{5}{1} - 1} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$x_3 = \frac{3}{4} - \frac{5\ln\frac{3}{4} - \frac{3}{4} + 2}{\frac{5}{3/4} - 1} = \frac{3}{4} - \frac{5\ln\frac{3}{4} + \frac{5}{4}}{\frac{17}{3}}$$

$$x_3 = \frac{3}{4} - \frac{3}{17}\left(5\ln\left(\frac{3}{4}\right) + \frac{5}{4}\right)$$
$$= \frac{3}{4} - \frac{3}{17}\cdot 5\left(\ln\left(\frac{3}{4}\right) + \frac{1}{4}\right)$$
$$= \frac{3}{4} - \frac{15}{17}\left(\ln\left(\frac{3}{4}\right) + \frac{1}{4}\right)$$

[5 marks]

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(c) A series is given as $\sum_{r=3}^{n} \frac{4}{r(r-2)}$.

(i) Show that $\left(\frac{1}{r-2} - \frac{1}{r}\right) = \frac{2}{r(r-2)}$ where $r \in \mathbb{N}$.

$\frac{1}{r-2} - \frac{1}{r}$

$\frac{r-(r-2)}{r(r-2)} = \frac{2}{r(r-2)}$

[2 marks]

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(ii) Hence, or otherwise, determine $\sum_{r=3}^{n} \frac{4}{r(r-2)}$ in terms of n.

$$\sum_{r=3}^{n} \frac{4}{r(r-2)} = 2\sum_{r=3}^{n} \frac{2}{r(r-2)} = 2\sum_{r=3}^{n}\left(\frac{1}{r-2} - \frac{1}{r}\right)$$

$$= 2\left[\left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) + \cdots \left(\cancel{\frac{1}{n-3}} - \frac{1}{n-1}\right) + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n}\right)\right]$$

$$= 2\left[1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right]$$

$$= 2\left[\frac{3}{2} - \frac{1}{n-1} - \frac{1}{n}\right]$$

[4 marks]

Total 20 marks

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SECTION C

Module 3

Answer this question.

3. (a) Determine the general solution of the differential equation $\frac{dy}{dx} + (\cos x)y = \cos x$.

$$\text{I.F.} = e^{\int \cos x \, dx} = e^{\sin x}$$

$$y e^{\sin x} = \int \cos x \, e^{\sin x} \, dx$$

let $u = \sin x$

$\frac{du}{dx} = \cos x$

$$\int \cancel{\cos x} \, e^{u} \frac{du}{\cancel{\cos x}} = \int e^{u} \, du$$

$$= e^{u} + c$$

$$= e^{\sin x} + c$$

$$y e^{\sin x} = e^{\sin x} + c$$

$$y = 1 + \frac{c}{e^{\sin x}}$$

[5 marks]

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(b) Two fair coins are tossed along with a fair, six-sided die.

(i) List the simple events of the possibility space.

	HH	HT	TH	TT
1	HH1	✓	✓	TT1
2	HH2	✓	✓	✓
3	HH3	HT3	✓	✓
4	HH4	✓	✓	TT4
5	HH5	✓	TH5	✓
6	HH6	✓	✓	TT6

[4 marks]

(ii) Determine the probability that at least one head or a number greater than 4 is observed.

P(at least 1 head OR a number greater than 4)

$= \frac{20}{24} = \frac{5}{6}$

[6 marks]

(c) **A** and **B** are two matrices such that det(**AB**) = –880 and det(**A**) = 88. Determine x, given that $\mathbf{B} = \begin{pmatrix} 1 & 3 & -2 \\ 2 & x & 1 \\ 3 & 1 & -1 \end{pmatrix}$.

$\det(AB) = \det A \det B$

$B = \begin{bmatrix} 1 & 3 & -2 \\ 2 & x & 1 \\ 3 & 1 & -1 \end{bmatrix} \Rightarrow \det B = (-x-1) - 3(-2-3) + (-2)(2-3x)$

$\det B = -x - 1 + 15 - 4 + 6x$

$= 5x + 10$

so $88 \times (5x+10) = -880$

$5x + 10 = -10$

$5x = -20$

$x = -4$

[5 marks]

Total 20 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.