FORM TP 2019309

MAY/JUNE 2019

CARIBBEAN EXAMINATIONS COUNCIL

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 2 – Paper 032

ANALYSIS, MATRICES AND COMPLEX NUMBERS

1 hour 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Each section consists of ONE question.
3. Answer ALL questions.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

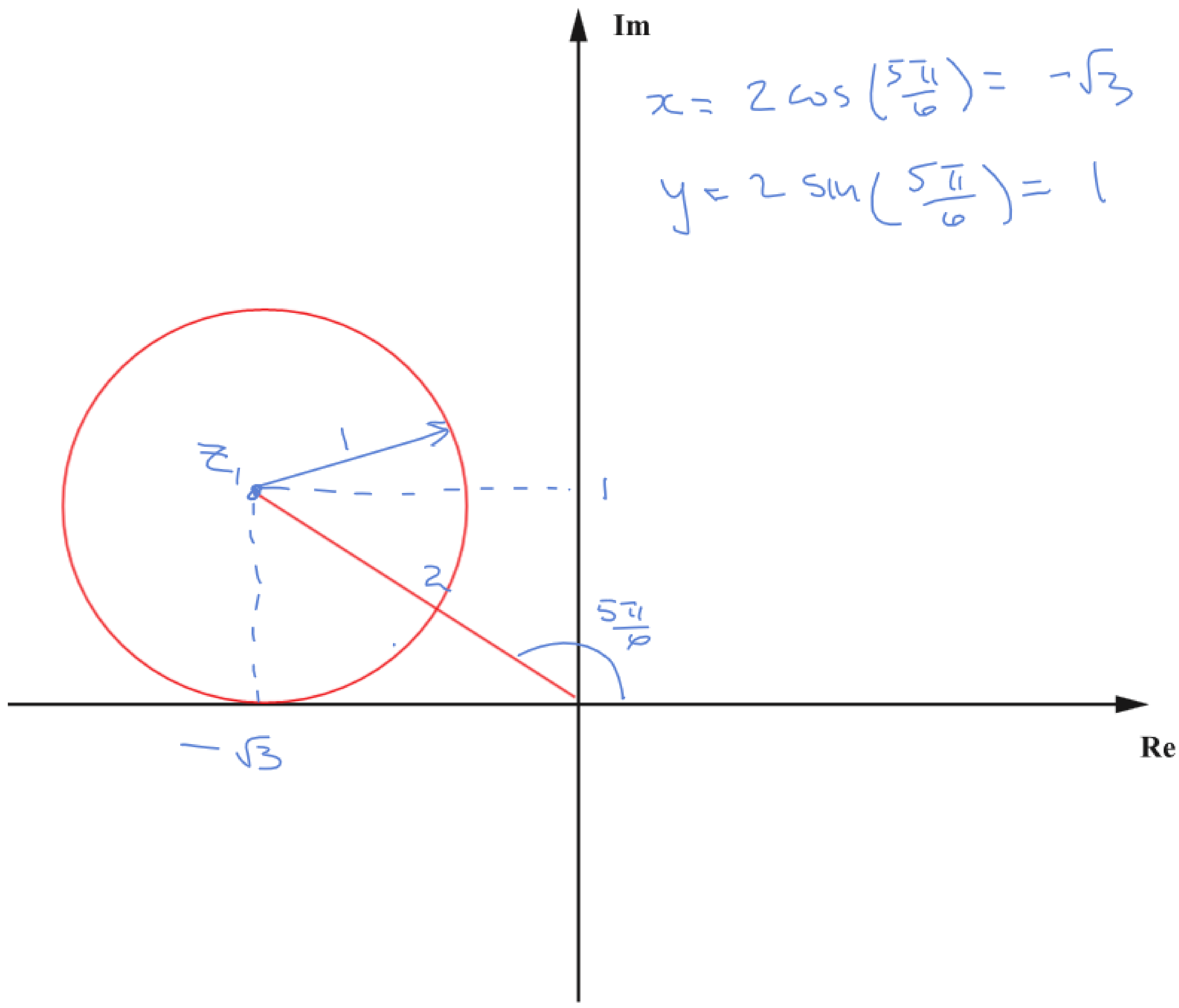
SECTION A

Module 1

Answer this question.

1. (a) A complex number z_1 is such that $|z_1| = 2$ and $\arg z_1 = \frac{5\pi}{6}$.

Use the Argand diagram below to answer (a) (i), (ii) and (iii).



(i) Determine the coordinates of z_1. **[3 marks]**

(ii) Connect z_1 to the Origin with a straight line and label the angle that represents $\arg z_1$. **[2 marks]**

(iii) Sketch the locus of the point z which moves in the complex plane such that $|z - z_1| = 1$. **[2 marks]**

(b) Use the trapezium rule, with four smaller intervals of equal width, to approximate

$$\int_0^2 \sqrt{3+x^2}\,.$$

interval $= \frac{2}{4} = \frac{1}{2} = 0.5$

$$A = \frac{0.5}{2}\left[f(0) + f(2) + 2(f(0.5) + f(1) + f(1.5))\right]$$

$$= \frac{0.5}{2}\left[1.732 + 2.631 + 2(1.803 + 2 + 2.291)\right]$$

$$= 0.25\left[4.363 + 2(6.094)\right] = 4.138$$

[6 marks]

(c) Determine $\int_0^1 \frac{dx}{\sqrt{9-x^2}}$. $= I$

when $x=0$ $\theta=0$
when $x=1$ $\theta=\sin^{-1}\frac{1}{3}$

let $x=3\sin\theta$
$dx=3\cos\theta\, d\theta$

$9-(3\sin\theta)^2 = 9-9\sin^2\theta = 9(1-\sin^2\theta)$

$\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = 3\cos\theta$

so $I = \int \frac{3\cos\theta}{3\cos\theta}\, d\theta = \int_0^{\sin^{-1}(1/3)} d\theta = \theta + c$

$= \sin^{-1}\left(\frac{1}{3}\right) - 0 = \sin^{-1}\left(\frac{1}{3}\right)$

[7 marks]

Total 20 marks

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SECTION B

Module 2

Answer this question.

2. (a) (i) Show that $\frac{5}{4r^2 - 1} = \frac{5}{2(2r-1)} - \frac{5}{2(2r+1)}$.

$$= \frac{5(2r+1) - 5(2r-1)}{2(2r-1)(2r+1)}$$

$$= \frac{10}{2(2r-1)(2r+1)} = \frac{5}{4r^2-1}$$

[3 marks]

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(ii) Hence, or otherwise, show that $\sum_{r=1}^{n} \frac{5}{4r^2 - 1} = \frac{5n}{2n+1}$.

$$\sum_{r=1}^{n} \frac{5}{4r^2-1} = \sum_{r=1}^{n} \frac{5}{2}\left(\frac{1}{2r-1} - \frac{1}{(2r+1)}\right)$$

$$= \frac{5}{2}\left[\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) \cdots \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)\right]$$

$$= \frac{5}{2}\left[1 - \frac{1}{2n+1}\right]$$

$$= \frac{5}{2}\left[\frac{2n}{2n+1}\right] = \frac{5n}{2n+1}$$

[5 marks]

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(b) An arithmetic progression is such that the fourth and tenth partial sums are $S_4 = -24$ and $S_{10} = 0$, respectively.

(i) Calculate the first term and the common difference.

$S_4 = \frac{4}{2}(2a + (4-1)d)$ $\quad$ $S_{10} = \frac{10}{2}(2a + 9d)$

$2(2a + 3d) = -24$ $\quad \Rightarrow \quad 2a + 3d = -12$

$5(2a + 9d) = 0$ $\quad\quad 2a + 9d = 0$

$\Rightarrow \quad d = 2$ (common difference)

$a = -9$ (first term)

[5 marks]

(ii) Hence, or otherwise, calculate the 15th term of the progression.

$u_{15} = -9 + 14(2)$
$= 19$

[2 marks]

(c) (i) Show that the equation $2e^x + x^2 - 3 = 0$ has a root, α, in the interval $(-2, -1)$.

Let $f(x) = 2e^x + x^2 - 3$
$f(x)$ is continuous
$f(-2) = 1.27$
$f(-1) = -1.26$
$\therefore$ according to the intermediate value theorem there is at least one root α in the interval $(-2, -1)$.

[3 marks]

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(ii) Apply linear interpolation once in the interval (–2, –1) to find an approximation to the root, α.

$$\alpha = \frac{-2(1.26) + -1(1.27)}{|-1.26| + |1.27|}$$

$$= -1.498$$

[2 marks]

Total 20 marks

SECTION C

Module 3

Answer this question.

3. (a) (i) To make three-letter codes, three letters are selected without replacement from the word TRAVEL and are written down in the order in which they are selected. How many three-letter codes may be formed?

$^6P_3 = 120$

[2 marks]

(ii) For a three-letter code to be legal, it must have at least one vowel. What is the probability that a legal word is formed on a single attempt?

Possible arrangements

AE + 1 other letter $= {}^4C_1 \times 3! = 4 \times 6 = 24$

A + 2 other letters $= {}^4C_2 \times 3! = 36$

E + 2 other letters $= {}^4C_2 \times 3! = 36$

Total number of legal words $= 96$

Probability $= \frac{96}{120}$

OR

no vowels i.e. only T R V L $= {}^4P_3 = 24$

so at least one vowel $= 120 - 24 = 96$

so Prob $= \frac{96}{120}$

[5 marks]

(b) A system of equations $\mathbf{Ax} = \mathbf{b}$ is given by

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 3 \end{pmatrix}.$$

(i) Calculate $|\mathbf{A}|$.

$|A| = 1(2+6) - 1(-4-3) + (-1)(-4+1)$

$= 8 + 7 + 3 = 18$

[3 marks]

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“*”Barcode Area”*”
Sequential Bar Code

(ii) Let the matrix $\mathbf{C} = \begin{pmatrix} 8 & 7 & -3 \\ 4 & -1 & 3 \\ 2 & -5 & -3 \end{pmatrix}$. Show that $\mathbf{C}^T\mathbf{A} - 18\mathbf{I} = 0$.

$$C^T = \begin{pmatrix} 8 & 4 & 2 \\ 7 & -1 & -5 \\ -3 & 3 & -3 \end{pmatrix}$$

$$C^T A = \begin{pmatrix} 8 & 4 & 2 \\ 7 & -1 & -5 \\ -3 & 3 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ 1 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$

$$18I = 18\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$

$$\therefore C^T A - 18I = 0$$

[4 marks]

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"*"Barcode Area"*"
Sequential Bar Code

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(iii) Hence, or otherwise, obtain $\mathbf{A}^{-1}$.

$$A^{-1} = \frac{1}{18}\begin{pmatrix} 8 & 4 & 2 \\ 7 & -1 & -5 \\ -3 & 3 & -3 \end{pmatrix}$$

[2 marks]

(iv) Solve the given system of equations for x, y and z.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 8 & 4 & 2 \\ 7 & -1 & -5 \\ -3 & 3 & -3 \end{pmatrix}\begin{pmatrix} 6 \\ -9 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

[4 marks]

Total 20 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.