C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 2 – Paper 032

ANALYSIS, MATRICES AND COMPLEX NUMBERS

1 hour 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Each section consists of ONE question.
3. Answer ALL questions.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

__**Examination Materials Permitted**__

Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

“*”Barcode Area”*”
Sequential Bar Code

SECTION A

Module 1

Answer this question.

1. (a) A point moves in the complex plane such that $|z - 3 - i| = 4$. A second point moves in the complex plane such that $|z - 4| = |z + 3|$.

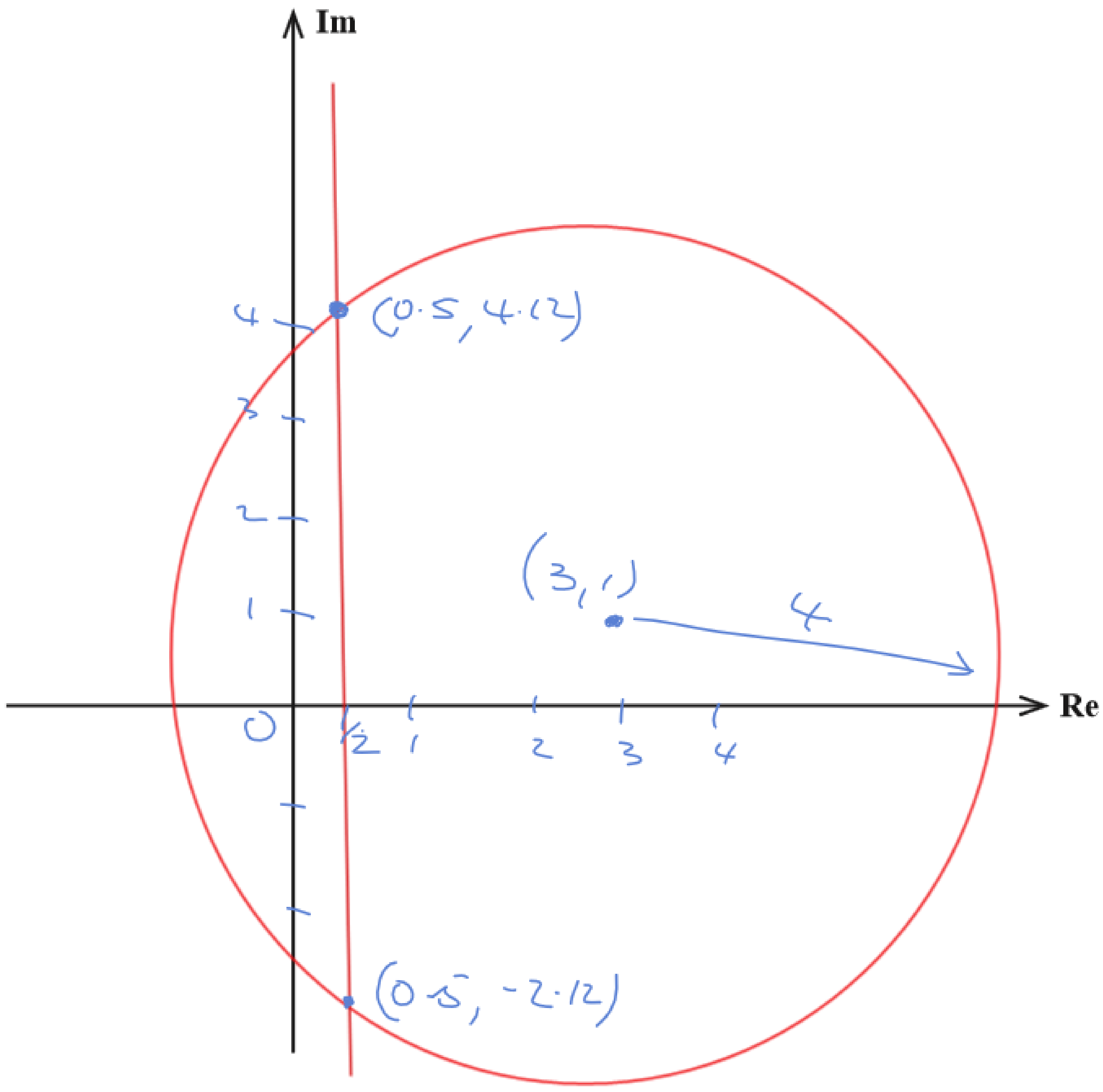
(i) Identify the loci of the points giving descriptions, if necessary.

Circle centre (3,1) and radius 4

The line $x = \frac{1}{2}$

[2 marks]

(ii) Using the axes below, sketch the loci of the two points on the same Argand diagram.



[4 marks]

(iii) Determine the points that satisfy the two loci.

$x = \frac{1}{2}$

equation of the circle is

$(x-3)^2 + (y-1)^2 = 16$

$(\frac{1}{2}-3)^2 + (y-1)^2 = 16$

$(-2.5)^2 + (y-1)^2 = 16$

$(y-1)^2 = 16 - (-2.5)^2$

$(y-1)^2 = 9.75$

$y-1 = \sqrt{9.75}$

$y = \pm\sqrt{9.75} + 1$

$y = 4.12$ or $y = -2.12$

So points are

$(0.5, 4.12)$ or $(0.5, -2.12)$

[4 marks]

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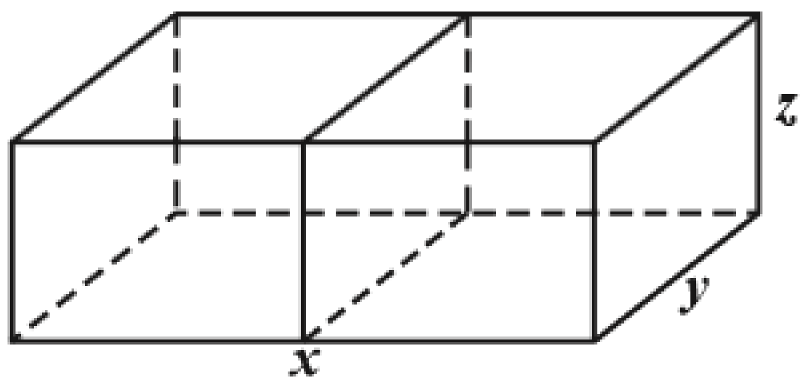
(b) The diagram below (**not drawn to scale**) shows an open rectangular box with a partition in the middle.



The dimensions of the box, measured in centimetres, are x, y, and z, and the volume of the box is 384 cm^3. The pieces from which the box is assembled are cut from a flat plank of wood with a TOTAL (one-sided) area given by

$$A = xy + \frac{768}{y} + \frac{1152}{x}.$$

Given that stationary values of a function $f(x, y)$ occur when both $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, and that a stationary value is a

- minimum when $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y} > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$ and $\frac{\partial^2 f}{\partial y^2} > 0$
- maximum when $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y} > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$ and $\frac{\partial^2 f}{\partial y^2} < 0$,

(i) show that a stationary value of A occurs when $x = 12$ and $y = 8$.

$\frac{\partial f}{\partial x} = y - \frac{1152}{x^2} = 0$ and $\frac{\partial f}{\partial y} = x - \frac{768}{y^2} = 0$

combining the equations $y = \frac{1152}{\left(\frac{768}{y^2}\right)^2}$

$y^3 = \frac{768^2}{1152} = 512$

$y = \sqrt[3]{512} = 8$

so $x = \frac{768}{8^2} = 12$

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(ii) Determine the nature of the stationary point of A that occurs at (12, 8).

$\frac{\partial^2 f}{\partial x^2} = \frac{1152(2)}{x^3}$ at $x = 12$ $\frac{\partial^2 f}{\partial x^2} = 1.33$

$\frac{\partial^2 f}{\partial y^2} = \frac{768\ (2)}{y^3}$ at $y = 8$ $\frac{\partial^2 f}{\partial y^3} = 3$

$\frac{\partial^2 f}{\partial x \partial y} = 1$

So $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y} = (1.33)(3) - 1 = 2.99 > 0$

so stationary point is a minimum

[3 marks]

Total 20 marks

SECTION B

Module 2

Answer this question.

2. (a) (i) Show that $\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{2}{4r^2-1}$.

$$\frac{2r+1-(2r-1)}{(2r-1)(2r+1)} = \frac{2}{4r^2-1}$$

[3 marks]

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(ii) Hence, or otherwise, show that $\sum_{r=1}^{n} \frac{2}{4r^2-1} = \frac{2n}{2n+1}$.

$$\sum_{1}^{n} \frac{2}{4r^2-1} = \sum_{1}^{n} \frac{1}{2r-1} - \frac{1}{2r+1}$$

$$= \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{7}}\right) + \cdots \left(\cancel{\tfrac{1}{2n-1}}\right) - \left(\frac{1}{2n+1}\right)$$

$$= 1 - \frac{1}{2n+1}$$

$$= \frac{2n+1-1}{2n+1} = \frac{2n}{2n+1}$$

[5 marks]

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(b) (i) Determine the binomial expansion of $(1 + \frac{1}{4}x)^5$.

$$\left(1+\frac{1}{4}x\right)^5 = 1 + \binom{5}{1}\left(\frac{1}{4}x\right) + \binom{5}{2}\left(\frac{1}{4}x\right)^2 + \binom{5}{3}\left(\frac{1}{4}x\right)^3 + \binom{5}{4}\left(\frac{1}{4}x\right)^4 + \left(\frac{1}{4}x\right)^5$$

$$= 1 + \frac{5}{4}x + \frac{5x^2}{8} + \frac{5x^3}{32} + \frac{5x^4}{256} + \frac{x^5}{1024}$$

[3 marks]

(ii) Hence, compute 1.025^5, correct to two decimal places.

$$1.025 = 1 + \frac{x}{4} \Rightarrow x = 0.1$$

$$1.025^5 = \left(1 + \frac{0.1}{4}\right)^5$$

$$= 1 + \frac{5(0.1)}{4} + \frac{5(0.1)^2}{8} + \frac{5(0.1)^3}{32} + \frac{5(0.1)^4}{256} + \frac{(0.1)^5}{1024}$$

$$= 1.13 \text{ (2 decimal places)}$$

[3 marks]

(c) Use the method of bisection to obtain an approximation of $\sqrt{3}$, correct to 2 decimal places.
Hint: Use the function $f(x) = x^2 - 3$ in the interval [1.5, 1.9].

$x^2 = 3 \Rightarrow x^2 - 3 = 0 \Rightarrow f(x) = x^2 - 3$

$f(1.5) = -0.75$ and $f(1.9) = 0.61$

a	b	x	f(x)
1.5	1.9	1.7	−0.11
1.7	1.9	1.8	0.24
1.7	1.8	1.75	0.0625
1.7	1.75	1.725	−0.02437
1.725	1.75	1.7375	0.018906
1.725	1.7375	1.73125	−0.00277

so $\sqrt{3} = 1.73$ (to 2 decimal places)

[6 marks]

Total 20 marks

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SECTION C

Module 3

Answer this question.

3. (a) A matrix, **A**, is given as $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$.

(i) Determine the inverse of the matrix **A**.

$$\det A = (12-9) - (6-3) + (3-2) = 1$$

$$\text{cof } A = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

[5 marks]

(ii) Determine a 3 × 1 matrix Y such that $A\begin{pmatrix}3\\-1\\2\end{pmatrix} = Y$.

$$\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 3 & 6\end{bmatrix}\begin{bmatrix}3\\-1\\2\end{bmatrix} = \begin{bmatrix}4\\7\\12\end{bmatrix}$$

[2 marks]

(iii) Hence, determine the 3 × 3 matrix B such that $BY = \begin{pmatrix}6\\-2\\4\end{pmatrix}$.

Let $X = \begin{bmatrix}3\\-1\\2\end{bmatrix} \Rightarrow 2X = \begin{bmatrix}6\\-2\\4\end{bmatrix}$ $\quad AX = Y$

$BY = 2X$

$BAX = 2X$

$\therefore \quad BA = 2$

$BAA^{-1} = 2A^{-1}$

$BI = 2A^{-1}$

$B = 2A^{-1}$

$$B = 2\begin{bmatrix}3 & -3 & 1\\-3 & 5 & -2\\1 & -2 & 1\end{bmatrix}$$

[3 marks]

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(b) A differential equation involving a learning curve $P(t)$ is given as

$$\frac{dP}{dt} = k\,(M - P(t))$$

where k is a positive constant, M is the maximum level of performance of which a learner is capable and t is the time.

Solve the differential equation by finding the general solution for $P(t)$.

$$\frac{dP}{dt} = kM - k\,P(t)$$

$$\frac{dP}{dt} + k\,P(t) = kM \qquad I.F. = e^{\int k\,dt} = e^{kt}$$

$$e^{kt}\,P(t) = \int e^{kt}\,km\,dt = me^{kt} + c$$

$$P(t) = m + ce^{-kt}$$

[4 marks]

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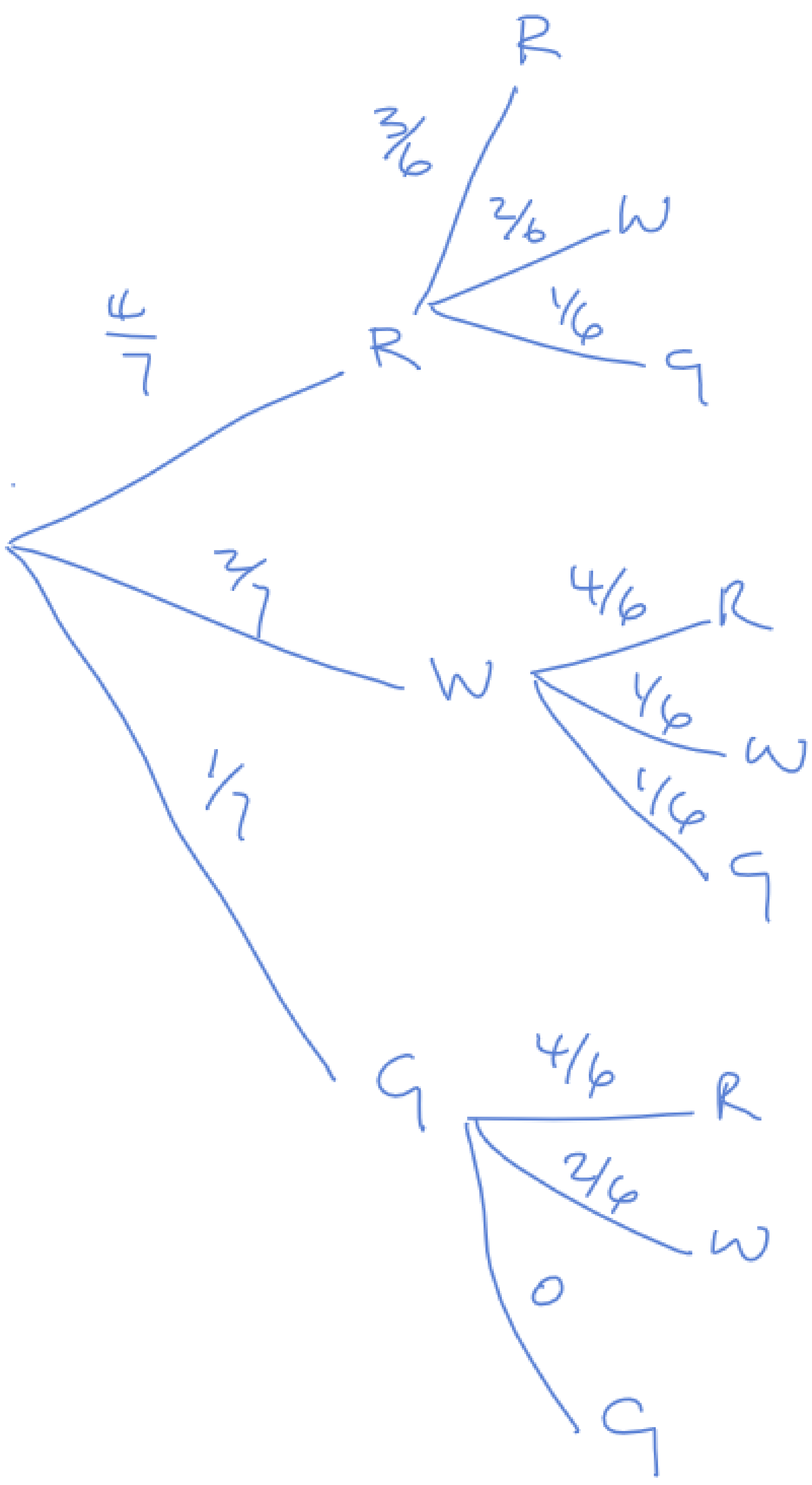
(c) A bag contains 4 red balls, 2 white balls and 1 green ball. Two balls are to be drawn at random from the bag without replacement.

(i) Use a tree diagram to represent the outcomes of the two draws and the corresponding probabilities.



[4 marks]

(ii) Hence, or otherwise, determine the probability that the second ball drawn is green.

P(second ball is green)

$= \frac{4}{7} \cdot \frac{1}{6} + \frac{2}{7} \cdot \frac{1}{6} + \frac{1}{7} \cdot 0 = \frac{6}{42} = \frac{1}{7}$

[2 marks]

Total 20 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.