CAPE 2017 Paper 2 SOLUTIONS

QUESTION 1

(a) (i) $\sim p \vee q$

(ii) $p \rightarrow \sim q$

(b) (i) $*$ is commutative since $2 * 1 = 1 * 2$

The table is symmetric about the diagonal hence * is commutative or showing that a*b =b*a for all a,b.

(ii) $1 * 3 = 1$

$2 * 3 = 2$

3*a=a*3 for all a in {1, 2,3,4} therefore 3 is the identity element of *.

$3 * 3 = 3$

$4 * 3 = 4$

(c) (i) $f(x) = ax^3 + 9x^2 - 11x + b$

$f(2) = 0$

$a(2)^3 + 9(2)^2 - 11(2) + b = 0$

$8a + b = -14$

$f(-2) = 12$

$a(-2)^3 + 9(-2)^2 - 11(-2) + b = 12$

$-8a + b = -46$

$2b = -60$

$b = -30$

$a = 2$

(ii) $f(x) = 2x^3 + 9x^2 - 11x - 30$

	x^3	x^2	x	constant
	2	9	−11	−30
	↓	−4	−26	−30
$x-2$	2	13	15	0

$f(x) = (x - 2)(2x^2 + 13x + 15)$

$f(x) = (x - 2)(2x + 3)(x + 5)$

$$x = -5, -\frac{3}{2}, 2$$

(d) Let $P_n: \sum_{r=1}^{n} 8r = 4n(n+1) \quad \forall\, n \in \mathbb{N}$

$P_1: 8(1) = 4(1)(1+1)$

$8 = 8$

Therefore P_1 is true.

Assume P_n is true for $n = k$

$$P_k: \sum_{r=1}^{k} 8r = 4k(k+1)$$

$$P_{k+1}: \sum_{r=1}^{k+1} 8r = 4(k+1)(k+2)$$

$P_{k+1} = P_k + (k+1)\text{term}$

$= 4k(k+1) + 8(k+1)$

$= (k+1)(4k+8)$

$= 4(k+1)(k+2)$

Therefore P_{k+1} is true $\forall$ P_k is true.

Hence by mathematical induction $8 + 11 + 24 + 32 + \cdots + 8n = 4n(n+1)$ for all $n \in \mathbb{N}$.

QUESTION 2

2. (a) (i)

$$\ln\left(\frac{a+b}{4}\right) = \frac{1}{2}(\ln a + \ln b)$$

$$\ln\left(\frac{a+b}{4}\right) = \frac{1}{2}(\ln ab)$$

$$\ln\left(\frac{a+b}{4}\right) = \ln(ab)^{\frac{1}{2}}$$

$$\frac{a+b}{4} = \sqrt{ab}$$

$$a + b = 4\sqrt{ab}$$

$$(a+b)^2 = 16ab$$

$$a^2 + 2ab + b^2 = 16ab$$

$$a^2 + b^2 = 14ab$$

Alternately

$$(a+b)^2 = a^2 + 2ab + b^2$$

Therefore

$$a^2 + b^2 = 14ab$$

$$a^2 + b^2 - 14ab = 0$$

$$a^2 + b^2 + 2ab - 16ab = 0$$

$$(a+b)^2 = 16ab$$

$$\frac{(a+b)^2}{16} = ab$$

$$\left(\frac{a+b}{4}\right)^2 = ab$$

$$\ln\left(\frac{a+b}{4}\right)^2 = \ln ab$$

$$2\ln\left(\frac{a+b}{4}\right) = \ln a + \ln b$$

$$\ln\left(\frac{a+b}{4}\right) = \frac{1}{2}(\ln a + \ln b)$$

(ii) $2^{-x} + 3(2^x) = 4$

$\frac{1}{2^x} + 3(2^x) = 4$

$\frac{1}{y} + 3y - 4 = 0$

$3y^2 - 4y + 1 = 0$

$(3y-1)(y-1) = 0$

$y = \frac{1}{3}, 1$

$2^x = \frac{1}{3}$

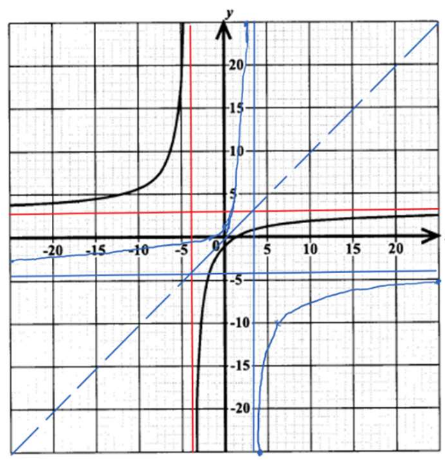
$\ln 2^x = \ln \frac{1}{3}$

$x = \frac{\ln\left(\frac{1}{3}\right)}{\ln 2} = -\frac{\ln 3}{\ln 2}$

$2^x = 1$

$x = 0$

(b)



(c) $\alpha + \beta + \gamma = -\frac{b}{a} = 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 3$

$\alpha\beta\gamma = -\frac{d}{a} = -2$

$x^3 - (\alpha\beta + \alpha\gamma + \beta\gamma)x^2 + \big((\alpha\beta)(\alpha\gamma) + (\alpha\beta)(\beta\gamma) + (\alpha\gamma)(\beta\gamma)\big)x - \big((\alpha\beta)(\alpha\gamma)(\beta\gamma)\big) = 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = 3$

$(\alpha\beta)(\alpha\gamma) + (\alpha\beta)(\beta\gamma) + (\alpha\gamma)(\beta\gamma)$

$= \alpha^2\beta\gamma + \beta^2\alpha\gamma + \gamma^2\alpha\beta$

$= (\alpha\beta\gamma)(\alpha + \beta + \gamma)$

$= (-2)(0)$

$= 0$

$(\alpha\beta)(\alpha\gamma)(\beta\gamma)$

$= (\alpha\beta\gamma)^2$

$= (-2)^2$

$= 4$

$x^3 - 3x^2 - 4 = 0$

QUESTION 3

3. (a) (i) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B - \sin A \sin B}$$

$$= \left(\frac{\sin A \cos B}{\cos A \cos B} + \frac{\sin B \cos A}{\cos A \cos B}\right) \div \left(\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}\right)$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

ALTERNATELY

$$\frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \left(\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}\right) \div \left(1 - \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)\right)$$

$$= \left(\frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B}\right) \div \left(\frac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}\right)$$

$$= \left(\frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B}\right) \times \left(\frac{\cos A \cos B}{\cos A \cos B - \sin A \sin B}\right)$$

$$= \frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \tan(A+B)$$

(ii) $\sin A = \frac{3}{5}$

$$\tan A = \frac{3}{4}$$

$$\cos B = -\frac{1}{2}$$

$$\tan B = -\sqrt{3}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\left(\frac{3}{4}\right) + (-\sqrt{3})}{1 - \left(\frac{3}{4}\right)(-\sqrt{3})}$$

$$= \left(\frac{3}{4} - \sqrt{3}\right) \div \left(1 + \frac{3\sqrt{3}}{4}\right)$$

$$= \left(\frac{3 - 4\sqrt{3}}{4}\right) \times \left(\frac{4 + 3\sqrt{3}}{4}\right)$$

$$= \frac{3 - 4\sqrt{3}}{4 + 3\sqrt{3}}$$

$$= \frac{(3 - 4\sqrt{3})(4 - 3\sqrt{3})}{(4 + 3\sqrt{3})(4 - 3\sqrt{3})}$$

$$= \frac{12 - 9\sqrt{3} - 16\sqrt{3} + 36}{16 - 27}$$

$$= \frac{48 - 25\sqrt{3}}{-11}$$

$$= -\frac{48}{11} + \frac{25\sqrt{3}}{11}$$

(b) $\sin^2\theta - 2\cos^2\theta + 3\cos\theta + 5 = 0$
$(1-\cos^2\theta) - 2\cos^2\theta + 3\cos\theta + 5 = 0$
$-3\cos^2\theta + 3\cos\theta + 6 = 0$
$\cos^2\theta - \cos\theta - 2 = 0$
$(\cos\theta - 2)(\cos\theta + 1) = 0$

$\cos\theta = -1$
$\theta = \pi$
$\theta = 3\pi$
By using the graph of $\cos\theta$.

(c) (i) $f(\theta) = 6\cos\theta + 8\sin\theta$
$r = \sqrt{6^2+8^2} = 10$
$\alpha = \tan^{-1}\left(\frac{6}{8}\right) = 36.87°$
$f(\theta) = 10\sin(\theta + 36.87°)$

(ii) $10\sin(\theta + 36.87°) = 2$
$\sin(\theta + 36.87°) = 0.2$
$RA = \sin^{-1}(0.2) = 11.54°$
$I: \theta + 36.87° = 11.54°$
$\theta = -25.33°$
$II: \theta + 36.87° = 180° - 11.54°$
$= 168.46°$
$\theta = 131.59°$
General solution:
$\theta = -25.33 + 360°n, \ \ n \in \mathbb{Z}$
$\theta = 131.69° + 360°n, n \in \mathbb{Z}$

QUESTION 4

(a) (i) $x^2 + y^2 - 4x + 2y - 2 = 0$

$(x-2)^2 + (y+1)^2 = 2 + 4 + 1$

$(x-2)^2 + (y+1)^2 = 7$

Centre $(2, -1)$

Radius of $C_2 =$
$\sqrt{(2-(-1))^2 + (-1-(-2))^2} = \sqrt{10}$

$C_2: (x-2)^2 + (y+1)^2 = 10$

(ii) $x^2 + y^2 - 4x + 2y + 2 = 0$

$x + 3y = 3 \quad \rightarrow \quad x = 3 - 3y$

$(3-3y)^2 + y^2 - 4(3-3y) + 2y - 2 = 0$

$9y^2 - 18y + 9 + y^2 - 12 + 12y + 2y - 2 = 0$

$10y^2 - 4y - 5 = 0$

$b^2 - 4ac$

$(-4)^2 - 4(10)(-5) > 0$

Since discriminant is positive the quadratic has 2 distinct roots therefore the L_1 cannot be a tangent to C_1.

(b) (i) $\overrightarrow{PQ} = -2i - 3j + 2k$

(ii) $r.n = a.n$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} . \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$$

$-2x - 3y + 2z = -2 + 6 + 8$

$-2x - 3y + 2z = 12$

(c) $L_1 = -i + j - 2k + \alpha(-2i + j - 3k)$

$L_2 = -2i + j - 4k + \beta(i - j + k)$

$-i + j - 2k - 2\alpha i + \alpha j - 3\alpha k = -2i + j - 4k + \beta i - \beta j + \beta k$

$(-1 - 2\alpha)i = (-2 + \beta)i$

$-1 - 2\alpha = -2 + \beta$

$2\alpha + \beta = 1 \quad (1)$

$(1 + \alpha)j = (1 - \beta)j$

$1 + \alpha = 1 - \beta$

$\alpha + \beta = 0 \quad (2)$

$(-2 - 3\alpha)k = (-4 + \beta)k$

$-2 - 3\alpha = -4 + \beta$

$3\alpha + \beta = 2 \quad (3)$

Solving (1) and (2) simultaneously

$2\alpha + \beta = 1$

$\alpha + \beta = 0$

$\alpha = 1$

$\beta = -1$

Checking in (3)

$3(1) + (-1) = 2$

Therefore L_1 and L_2 intersect.

(ii) $L_1 = -i + j - 2k - 2i + j - 3k$ since $\alpha = 1$

$L_1 = -3i + 2j - 5k$

When $\beta = -1$

$L_2 = -2i + j - 4k - i + j - k$

$L_2 = -3i + 2j - 5k$

Point of intersection is $-3i + 2j - 5k$

QUESTION 5

(a) $\lim_{x\to 1} \frac{x^5 - 1}{x - 1} = k$

$$\lim_{x\to 1} \frac{(x-1)(x^4 + x^3 + x^2 + x + 1)}{x - 1} = k$$

$\lim_{x\to 1} x^4 + x^3 + x^2 + x + 1 = k$

$1^4 + 1^3 + 1^2 + 1 + 1 = k$

$5 = k$

(b) (i) $x = 5t + 3,\ y = t^3 - t^2 + 2$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = 3t^2 - 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{3t^2 - 2t}{5}$$

(ii) $\frac{dy}{dx} = 0$

$$3t^2 - 2t = 0$$

$$t(3t - 2) = 0$$

$$t = 0, \frac{2}{3}$$

When $t = 0$

$$x = 5(0) + 3 = 3$$

$$y = 0^3 - 0^2 + 2 = 2$$

$(3, 2)$

When $t = \frac{2}{3}$

$$x = 5\left(\frac{2}{3}\right) + 3 = \frac{19}{3}$$

$$y = \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 + 2 = \frac{50}{27}$$

$\left(\frac{19}{3}, \frac{50}{27}\right)$

(c) (i) (a) $y = \sqrt{2 + 2x^2} = (2 + 2x^2)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(2 + 2x^2)^{-\frac{1}{2}}(4x)$$

$$\frac{dy}{dx} = 2x(2 + 2x^2)^{-\frac{1}{2}}$$

$$y\frac{dy}{dx} - 2x$$

$$= (2 + 2x^2)^{\frac{1}{2}}\, 2x(2 + 2x^2)^{-\frac{1}{2}} - 2x$$

$$= 2x - 2x$$

$$= 0$$

(b) $\frac{d^2y}{dx^2} = 2(2 + 2x^2)^{-\frac{1}{2}} + 2x\left[-\frac{1}{2}(2 + 2x^2)^{-\frac{3}{2}}(4x)\right]$

$$\frac{d^2y}{dx^2} = 2(2 + 2x)^{-\frac{1}{2}} - 4x^2(2 + 2x^2)^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} - \frac{4}{y^3}$$

$$= 2(2 + 2x^2)^{-\frac{1}{2}} - 4x^2(2 + 2x^2)^{-\frac{3}{2}} - 4\left[(2 + 2x^2)^{-\frac{1}{2}}\right]^3$$

$$= 2(2 + 2x^2)^{-\frac{1}{2}} - 4x^2(2 + 2x^2)^{-\frac{3}{2}} - 4(2 + 2x^2)^{-\frac{3}{2}}$$

$$= 2(2 + 2x^2)^{-\frac{3}{2}}[2 + 2x^2 - 2x^2 - 2]$$

$$= 2(2 + 2x^2)^{-\frac{3}{2}}(0)$$

$$= 0$$

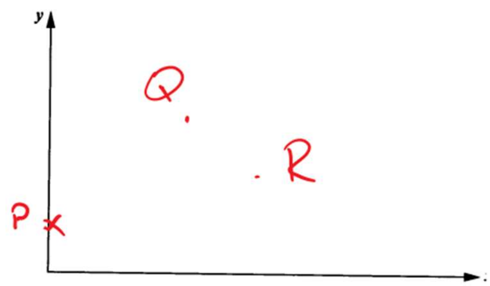
(ii) $\frac{d^2y}{dx^2}$ when $x = 0$

$$= 2\big(2 + 2(0)\big)^{-\frac{1}{2}} - 4(0)^2(2 + 2(0)^2)^{-\frac{3}{2}}$$

$$= \sqrt{2}$$

QUESTION 6

(a) (i)



(ii) Equation of PQ

$$m = \frac{3-1}{3-0} = \frac{2}{3}$$

$$y = \frac{2}{3}x + 1$$

Equation of QR

$$m = \frac{3-2}{3-4} = -1$$

$$y = mx + c \quad (3,3)\ m = -1$$

$$3 = -1(3) + c$$

$$c = 6$$

$$y = -x + 6$$

Equation of PR

$$m = \frac{2-1}{4-0} = \frac{1}{4}$$

$$y = \frac{1}{4}x + 1$$

(iii) $\int_0^3 \frac{2}{3}x + 1\ dx + \int_3^4 -x + 6\ dx - \int_0^4 \frac{1}{4}x + 1\ dx$

$$= \left[\frac{2}{3}\left(\frac{x^2}{2}\right) + x\right]_0^3 + \left[-\frac{x^2}{2} + 6x\right]_3^4 - \left[\frac{1}{4}\left(\frac{x^2}{2}\right) + x\right]_0^4$$

$$= \left[\left(\frac{(3)^2}{3} + 3\right) - \left(\frac{0^2}{3} + 0\right)\right] + \left[\left(-\frac{4^2}{2} + 6(4)\right) - \left(-\frac{3^2}{2} + 6(3)\right)\right] - \left[\left(\frac{4^2}{8} + 4\right) - \left(\frac{0^2}{8} + 0\right)\right]$$

$$= \left[(6-0) + \left(16 - \frac{27}{2}\right) - (6-0)\right]$$

$$= \frac{5}{2}$$

(c) $\int_{-1}^{3}[3f(x)+g(x)]\,dx=5$

and $\int_{-1}^{3}[5f(x)-2g(x)]\ dx=1$

$$\int_{-1}^{3}[3f(x)+g(x)]\ dx=5$$

$$3\int_{-1}^{3}f(x)\ dx+\int_{-1}^{3}g(x)\ dx=5$$

$$5\int_{-1}^{3}f(x)\,dx-2\int_{-1}^{3}g(x)\ dx=1$$

Let $\int_{-1}^{3}f(x)\ dx=m$ and $\int_{-1}^{3}g(x)\ dx=n$

$3m+n=5$ (1)

$5m-2n=1$ (2)

(1) × 2: $6m+2n=10$

(2) $5m-2n=1$

$11m=11$

$m=1$

$n=2$

$$\int_{-1}^{3}f(x)\ dx=1$$

$$\int_{-1}^{3}g(x)\ dx=2$$