TEST CODE **02234032**

FORM TP 2016283

MAY/JUNE 2016

C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 2 – Paper 032

ANALYSIS, MATRICES AND COMPLEX NUMBERS

1 hour 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Each section consists of ONE question.
3. Answer ALL questions.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra lined page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

“*”Barcode Area”*”
Sequential Bar Code

SECTION A

Module 1

Answer this question.

1. (a) Find the equations of the tangents to the curve $y = \frac{(x-1)}{(x+1)}$ that are parallel to the line $x - 2y = 1$.

$x - 2y = 1$

$x - 1 = 2y$

$\frac{1}{2}x - \frac{1}{2} = y$

$y = \frac{x-1}{x+1}$

$\frac{dy}{dx} = \frac{1(x+1) - (x-1)(1)}{(x+1)^2}$

$= \frac{2}{(x+1)^2}$

$\frac{1}{2} = \frac{2}{(x+1)^2}$

$(x+1)^2 = 4$

$x^2 + 2x + 1 = 4$

$x^2 + 2x - 3 = 0$

$(x - 1)(x + 3) = 0$

$x = 1$ $\quad$ $x = -3$

$y = 0$ $\quad$ $y = 2$

$(1, 0)$ $\quad$ $(-3, 2)$

$y = mx + c$ $\quad$ $y = mx + c$

$0 = \frac{1}{2}(1) + c$ $\quad$ $2 = \frac{1}{2}(-3) + c$

$-\frac{1}{2} = c$ $\quad$ $\frac{7}{2} = c$

$y = \frac{1}{2}x - \frac{1}{2}$ $\quad$ $y = \frac{1}{2}x + \frac{7}{2}$

[8 marks]

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(b) (i) Show that

$$\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx.$$

$u = x^n$ $v = -\cos x$

$du = nx^{n-1}$ $dv = \sin x$

$\int x^n \sin x \, dx = -x^n \cos x - \int n x^{n-1} (-\cos x) \, dx$

$= -x^n \cos x + n \int x^{n-1} \cos x \, dx$

[3 marks]

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(ii) Hence, or otherwise, given that

$$\int x^n \cos x\, dx = x^n \sin x - n \int x^{n-1} \sin x\, dx,$$

calculate $\int_0^{\frac{\pi}{2}} x^3 \sin x\, dx$.

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3\int x^2 \cos x\, dx$$

$$= -x^3 \cos x + 3\left[x^2 \sin x - 2\int x \sin x\, dx\right]$$

$$= -x^3 \cos x + 3x^2 \sin x - 6\int x \sin x\, dx$$

$$= -x^3 \cos x + 3x^2 \sin x - 6\left[-x \cos x + \int \cos x\, dx\right]$$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\int \cos x\, dx$$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$$

$$\int_0^{\pi/2} x^3 \sin x\, dx = \left[-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x\right]_0^{\pi/2}$$

$$= \left[-\left(\frac{\pi}{2}\right)^3 \cos \frac{\pi}{2} + 3\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + 6\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2}\right) - 6 \sin\left(\frac{\pi}{2}\right)\right]$$

$$= \left[\frac{3\pi^3}{4} - 6\right]$$

[5 marks]

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(c) Find the derivative of

$$x^2 \tan^{-1} e^{2x} + ln\, x^2$$

with respect to x for the domain $x > 0$.

$$y = x^2 \tan^{-1} e^{2x} + \ln x^2$$

$$\frac{dy}{dx} = 2x \tan^{-1} e^{2x} + x^2 \frac{2e^{2x}}{1+(e^{2x})^2} + \frac{2x}{x^2}$$

$$= 2x \tan^{-1} e^{2x} + \frac{2x^2 e^{2x}}{1+e^{4x}} + \frac{2}{x}$$

$$\boxed{\frac{d}{dx} \tan^{-1} u = \frac{u'}{1+u^2}}$$

[4 marks]

Total 20 marks

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SECTION B

Module 2

Answer this question.

2. (a) Find

$$\sum_{n=1}^{\infty}\left(\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right).$$

$$\sum_{n=1}^{K}\left(\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right)$$

$n=1$	$\sin\left(\frac{1}{1}\right)-\sin\left(\frac{1}{1+1}\right)$	$\sin 1 - \cancel{\sin\frac{1}{2}}$
$n=2$	$\sin\left(\frac{1}{2}\right)-\sin\left(\frac{1}{2+1}\right)$	$\cancel{\sin\frac{1}{2}} - \cancel{\sin\frac{1}{3}}$
$n=3$	$\sin\left(\frac{1}{3}\right)-\sin\left(\frac{1}{3+1}\right)$	$\cancel{\sin\frac{1}{3}} - \cancel{\sin\frac{1}{4}}$
$n=k-1$	$\sin\left(\frac{1}{k-1}\right)-\sin\left(\frac{1}{k-1+1}\right)$	$\cancel{\sin\left(\frac{1}{k-1}\right)} - \cancel{\sin\frac{1}{k}}$
$n=k$	$\sin\left(\frac{1}{k}\right)-\sin\left(\frac{1}{k+1}\right)$	$\cancel{\sin\left(\frac{1}{k}\right)} - \sin\left(\frac{1}{k+1}\right)$

$$\sum_{n=1}^{K}\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)=\sin 1-\sin\left(\frac{1}{k+1}\right)$$

$$\therefore \sum_{n=1}^{\infty}\left(\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right)=\sin 1-\sin(0)$$

$\boxed{\text{As } n\to\infty,\ \frac{1}{n+1}\to 0}$

$$=\sin 1$$

[6 marks]

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(b) (i) Express the number $0.\overline{15} = 0.1515151515...$ as a geometric series with the first term $a = \frac{15}{100}$.

$0.\overline{15} = 0.1515151515...$

$= \frac{15}{100} + \frac{15}{1\,000} + \frac{15}{1\,000\,000}$

$= \frac{15}{10^2} + \frac{15}{10^4} + \frac{15}{10^6}$

$a = \frac{15}{100}$ $r = \frac{1}{10^2}$

[4 marks]

(ii) Hence, express $0.\overline{15}$ as a ratio of integers.

$S_\infty = \frac{a}{1-r}$

$= \frac{\frac{15}{100}}{1 - \frac{1}{10^2}}$

$= \frac{5}{33}$

[2 marks]

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(c) (i) Show that the equation

$$f(x) = e^{x} \sin x - 2x$$

has a root in the interval $0.5 \leq x \leq 1$.

$f(x) = e^{x} \sin x - 2x$
$f(0.5) = -0.2096$
$f(1) \quad = 0.2874$

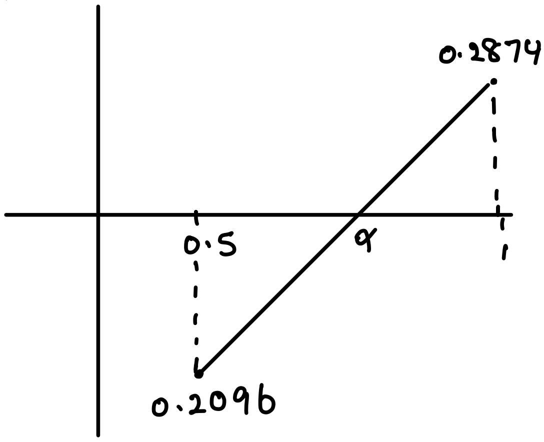
$f(x)$ is continuous on the interval $0.5 \leq x \leq 1$
$f(0.5) \times f(1) < 0$
By the Intermediate Value Theorem there is some c in $0.5 \leq x \leq 1$ such that $f(c) = 0$

[3 marks]

(ii) Use linear interpolation to approximate the root of f in the interval $0.5 \leq x \leq 1$, correct to 1 decimal place.

$$\frac{1-\alpha}{\alpha-0.5} = \frac{0.2874}{0.2096}$$

$$0.2096 - 0.2096\alpha = 0.2874\alpha - 0.1437$$
$$0.3533 = 0.497\alpha$$
$$0.7 = \alpha$$



[5 marks]

Total 20 marks

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SECTION C

Module 3

Answer this question.

3. (a) A farmer has 300 acres of land on which THREE crops, x, y and z are to be cultivated. The costs of cultivating x, y and z are, respectively, \$30, \$40 and \$50 per acre and the farmer has a total of \$11000 to spend on cultivation.

For each acre of crop x, y and z, 10, 15 and 40 labour hours, respectively, are required. A maximum of 6000 labour hours are available.

(i) Represent the information given with a system of linear equations.

$x + y + z = 300$ ①

$\$30x + \$40y + \$50z = \$11\,000$

$3x + 4y + 5z = 1100$ ②

$10x + 15y + 40z = 6000$

$2x + 3y + 8z = 1200$ ③

[3 marks]

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(ii) By converting the equations into matrix form, determine how many acres per crop the farmer may cultivate.

$x + y + z = 300$
$3x + 4y + 5z = 1100$
$2x + 3y + 8z = 1200$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 300 \\ 3 & 4 & 5 & 1100 \\ 2 & 3 & 8 & 1200 \end{array}\right)$$

$$\begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 300 \\ 0 & 1 & 2 & 200 \\ 0 & 1 & 6 & 600 \end{array}\right)$$

$$R_3 - R_2 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 300 \\ 0 & 1 & 2 & 200 \\ 0 & 0 & 4 & 400 \end{array}\right)$$

From Row 3
$4z = 400$
$z = 100$

From Row 2
$y + 2z = 200$
$y = 0$

From Row 1
$x + y + z = 300$
$x = 200$

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[8 marks]

(b) The vibration of a spring with a mass of 4 kg attached is described by the differential equation

$$4\frac{d^2x}{dt} + 400x = 0,$$

where x metres is the displacement of the mass at time t seconds.

The initial displacement of the mass is 0.25 m and the initial velocity is 0 (i.e. $x'(0) = 0$). Determine an expression for the displacement, x metres, of the mass at time t seconds.

$4\frac{d^2x}{dt^2} + 400x = 0$

$4u^2 + 400 = 0$

$u^2 + 100 = 0$

$u^2 = -100$

$u = \pm 10i$

$x = A\sin(10t) + B\cos(10t)$

when $t = 0$, $x = 0.25$

$0.25 = A\sin(0) + B\cos(0)$

$0.25 = B$

$x' = 10A\cos(10t) - 10B\sin(10t)$

$x'(0) = 0$

$0 = 10A\cos(0) - 10B\sin(0)$

$0 = A$

$x = 0.25\cos(10t)$

[9 marks]

Total 20 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.