TEST CODE **02134032**

# FORM TP 2016280

MAY/JUNE 2016

**C A R I B B E A N   E X A M I N A T I O N S   C O U N C I L**

**CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®**

**PURE MATHEMATICS**

**UNIT 1 – Paper 032**

**ALGEBRA, GEOMETRY AND CALCULUS**

***1 hour 30 minutes***

**READ THE FOLLOWING INSTRUCTIONS CAREFULLY.**

1. This examination paper consists of THREE sections.
2. Each section consists of ONE question.
3. Answer ALL questions.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra lined page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

**<u>Examination Materials Permitted</u>**

Mathematical formulae and tables (provided) – **Revised 2012**
Mathematical instruments
Silent, non-programmable, electronic calculator

**DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.**





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Sequential Bar Code

## SECTION A

### Module 1

**Answer this question.**

**1.** (a) (i) Using the substitution $y = e^x$, or otherwise, solve the equation $e^{2x} - 21e^x - 100 = 0$.

$y^2 - 21y - 100 = 0$

$(y + 4)(y - 25) = 0$

$y = -4 \quad y = 25$

$e^x = -4 \quad e^x = 25$

Invalid $\quad \ln e^x = \ln 25$

$x = \ln 25$

**[6 marks]**

(ii) Solve the inequality $|x + 5| \geq |3x + 2|$.

$x+5 \geqslant 3x+2$

$3 \geqslant 2x$

$\frac{3}{2} \geqslant x$

$x+5 \leq -(3x+2)$

$x+5 \leq -3x-2$

$4x \leq -7$

$x \leq -\frac{7}{4}$

**[4 marks]**

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(b) Let **p** and **q** be two propositions. Prove that $(\mathbf{p} \vee \mathbf{q}) \wedge \sim \mathbf{p} \equiv \sim \mathbf{p} \wedge \mathbf{q}$.

$(p \vee q) \wedge \sim p$

$= \sim p \wedge (p \vee q)$ Commutative

$= (\sim p \wedge p) \vee (\sim p \wedge q)$ Distributive

$= 0 \vee (\sim p \wedge q)$ Complement

$= \sim p \wedge q$ Absorption

**[6 marks]**

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(c) Use a direct proof to show that the sum of two odd numbers is even.

Let $2a+1$ be one odd number $a \in \mathbb{Z}$

" $2b+1$ " the second odd number $b \in \mathbb{Z}$

$2a+1+2b+1$

$= 2a+2b+2$

$= 2(a+b+1)$

$\therefore$ The sum of 2 odd numbers is even.

**[4 marks]**

**Total 20 marks**

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**SECTION B**

**Module 2**

**Answer this question.**

**2.** (a) (i) Express $\cos\theta - \sqrt{3}\sin\theta$ in the form $r\cos(\theta + \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$.

$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$\cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \frac{\pi}{3}\right)$

**[4 marks]**

(ii) Hence, or otherwise, find the general solution of the equation

$\cos\theta - \sqrt{3}\sin\theta = 0.$

$2\cos\left(\theta + \frac{\pi}{3}\right) = 0$

$\cos\left(\theta + \frac{\pi}{3}\right) = 0$

when $x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\cos x = 0$

$\therefore \theta + \frac{\pi}{3} = \frac{\pi}{2}$ $\quad\quad \theta + \frac{\pi}{3} = \frac{3\pi}{2}$

$\theta = \frac{\pi}{6}$ $\quad\quad \theta = \frac{7\pi}{6}$

General Solutions : $\theta = \begin{cases} \frac{\pi}{6} + 2n\pi \\ \frac{7\pi}{6} + 2n\pi \end{cases}$ $n \in \mathbb{Z}$

**[4 marks]**

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(b) Two lines $L_1$ and $L_2$ are defined parametrically by

$$L_1\text{:}\ \ x = 1 + t,\ \ y = 2 + 4t,\ \ z = -t$$

and

$$L_2\text{:}\ \ x = 2u,\ \ y = 3 + u,\ \ z = -2 + 3u$$

where $t$ and $u$ are scalars.

(i) Find the vector equations of $L_1$ and $L_2$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$$

$$L_1\text{:}\ r = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} + u\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$L_2\text{:}\ r = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} + u\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

**[3 marks]**

(ii) Hence, or otherwise, determine whether the lines intersect, are parallel or are skewed.

**Show ALL working to support your answer.**

$$\begin{pmatrix}1\\2\\0\end{pmatrix} + t\begin{pmatrix}1\\4\\-1\end{pmatrix} = \begin{pmatrix}0\\3\\-2\end{pmatrix} + u\begin{pmatrix}2\\1\\3\end{pmatrix}$$

$1+t = 0+2u$ ① $\rightarrow t = 2u-1$

$2+4t = 3+u$ ②

$0-t = -2+3u$ ③

sub ① into ②

$2+4(2u-1) = 3+u$

$2+8u-4 = 3+u$

$7u = 5$

$u = \frac{5}{7}$

sub ① into ③

$-(2u-1) = -2+3u$

$-2u+1 = -2+3u$

$3 = 5u$

$\frac{3}{5} = u$

The lines are not parallel since $\begin{pmatrix}1\\4\\-1\end{pmatrix}$ is not a multiple of $\begin{pmatrix}2\\1\\3\end{pmatrix}$. There is no solution to the system of equations so the lines are skewed.

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**[9 marks]**

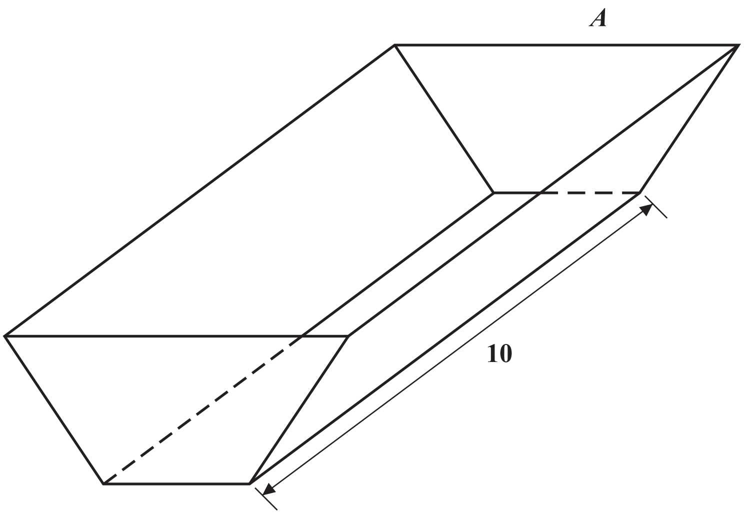
**Total 20 marks**

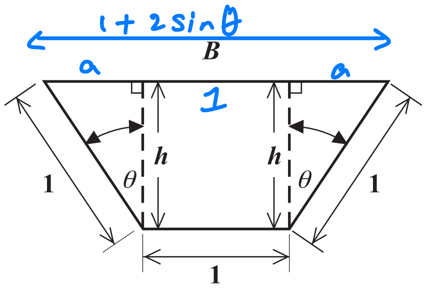
## SECTION C

### Module 3

**Answer this question.**

3. (a) Diagram $A$ below, **not drawn to scale**, shows the design of a trough. The cross-section of the trough, which has the shape of a trapezium, is shown in Diagram $B$. All lengths are in metres.




The trough must be made using the dimensions shown, but the angle $\theta$ may vary.

$\cos\theta = \frac{h}{1} \Rightarrow h = \cos\theta$

$\sin\theta = \frac{a}{1} \Rightarrow a = \sin\theta$

$\text{Area} = \frac{\cos\theta}{2}(1 + 1 + 2\sin\theta)$

$\text{Volume} = 10\left[\frac{\cos\theta}{2}(2 + 2\sin\theta)\right]$

$= 10\cos\theta(1 + \sin\theta)$

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(i) Given that the volume, $V$, is the product of the cross-sectional area and the length of the trough, show that

$$V = 10\cos\theta\,(1 + \sin\theta)$$

**[5 marks]**

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(ii) Hence, or otherwise, determine the MAXIMUM possible volume of the trough.

$V = 10\cos\theta(1+\sin\theta)$

$\frac{dV}{d\theta} = -10\sin\theta(1+\sin\theta) + 10\cos\theta(\cos\theta)$

Maximum occurs when $\frac{dV}{d\theta} = 0$ $\rightarrow$ $0 = -10\sin\theta - 10\sin^2\theta + 10\cos^2\theta$

$10\cos^2\theta - 10\sin^2\theta - 10\sin\theta = 0$

$10(1-\sin^2\theta) - 10\sin^2\theta - 10\sin\theta = 0$

$10 - 10\sin^2\theta - 10\sin^2\theta - 10\sin\theta = 0 \quad \div(-10)$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2}$ $\qquad$ $\sin\theta = -1$

$R.A = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ $\qquad$ $\theta = \frac{3\pi}{2}$

$I: \theta = \frac{\pi}{6}$

$II: \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

sub in each value of $\theta$ to determine maximum volume

when $\theta = \frac{\pi}{6}$

$V = 10\cos\left(\frac{\pi}{6}\right)\left(1+\sin\left(\frac{\pi}{6}\right)\right) = 13$ $\leftarrow$ Maximum Volume

when $\theta = \frac{5\pi}{6}$

$V = 10\cos\left(\frac{5\pi}{6}\right)\left(1+\sin\left(\frac{5\pi}{6}\right)\right) = -13$

when $\theta = \frac{3\pi}{2}$

$V = 10\cos\left(\frac{3\pi}{2}\right)\left(1+\sin\left(\frac{3\pi}{2}\right)\right) = 0$

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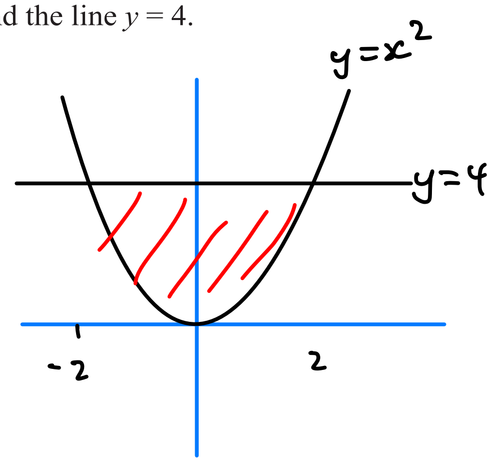
**[10 marks]**

(b) Calculate the area enclosed between the curve $y = x^2$ and the line $y = 4$.




$y = x^2$

$y = 4$

$x^2 = 4$

$x = \pm 2$

$$\text{Area} = \int_{-2}^{2} 4\,dx - \int_{-2}^{2} x^2\,dx$$

$$= \int_{-2}^{2} 4 - x^2\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left[4(2) - \frac{2^3}{3}\right] - \left[4(-2) - \frac{(-2)^3}{3}\right]$$

$$= \frac{16}{3} - \left[-\frac{16}{3}\right]$$

$$= \frac{32}{3}$$

**[5 marks]**

**Total 20 marks**

**END OF TEST**

**IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.**