HARRISON COLLEGE INTERNAL EXAMINATION, March 2020
CARIBBEAN ADVANCED PROFICIENCY EXAMINATION

SCHOOL BASED ASSESSMENT

PURE MATHEMATICS
UNIT 2 - TEST 1

TIME: 1 Hour & 20 minutes

This examination paper consists of 2 printed pages.
The paper consists of 3 questions.
The maximum mark for this examination is 60.

INSTRUCTIONS TO CANDIDATES

1. Write your name clearly on each sheet of paper used.
2. Answer **ALL** questions.
3. Number your questions carefully and do **NOT** write your solutions to different questions beside one another.
4. Unless otherwise stated in the question, any numerical answer that is not exact, **MUST** be written correct to three (3) significant figures.

EXAMINATION MATERIALS ALLOWED

1. Mathematical formulae
2. Electronic calculator (non-programmable, non-graphical)

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1. (a) The complex numbers z and w are given by $z = 2 + 3i$ and $w = -1 - 6i$ respectively. Find.

(i) $2z - w$ [1]

$= 2(2+3i) - (-1-6i)$

$= 4 + 6i + 1 + 6i$

$= 5 + 12i$ ✓

(ii) $|2z - w|$ [2]

$= \sqrt{5^2 + 12^2}$ ✓ F.T.

$= \sqrt{169}$

$= 13$ ✓

(iii) $\arg(2z - w)$ [2]

$= \tan^{-1}\left(\frac{12}{5}\right)$ ✓ FT

$= 1.17^c$ or 67.4° ✓

(iv) $\frac{z}{w}$ giving your answer in the form $x + iy$ [4]

$= \frac{2+3i}{-1-6i} \cdot \frac{-1+6i}{-1+6i}$ ✓

$= \frac{-2 - 3i + 12i + 18i^2}{1 - 36i^2}$ ✓ ✓

$= \frac{-20 + 9i}{37}$

$= -\frac{20}{37} + \frac{9}{37}i$ ✓

(b) (i) Express $\sin n\theta$ and $\cos n\theta$ in terms of $e^{in\theta}$ and $e^{-in\theta}$. [2]

$$\cos n\theta + i\sin n\theta = e^{in\theta}$$

$$\cos n\theta - i\sin n\theta = e^{-in\theta}$$

$$\sin n\theta = \frac{1}{2i}\left(e^{in\theta} - e^{-in\theta}\right) \checkmark$$

$$\cos n\theta = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right) \checkmark$$

(ii) Hence show that

$$sin^4\theta = \frac{1}{8}\,(cos4\theta - 4cos2\theta + 3) \qquad [5]$$

$$\sin^4\theta = (\sin\theta)^4 = \left[\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right]^4$$

$$= \frac{1}{16}\Big[(e^{i\theta})^4 + 4(e^{i\theta})^3(-e^{-i\theta}) + 6(e^{i\theta})^2(-e^{i\theta})^2$$

$$+ 4e^{i\theta}(-e^{i\theta})^3 + (-e^{-i\theta})^4\Big]$$

$$= \frac{1}{16}\left[\left(e^{i4\theta} + e^{-i4\theta}\right) + 4\left(e^{i2\theta} + e^{-i2\theta}\right) + 6\right]$$

$$= \frac{1}{16}\left[2\cos 4\theta + 8\cos 2\theta + 6\right]$$

$$= \frac{2}{16}\left[\cos 4\theta + 4\cos 2\theta + 3\right]$$

$$= \frac{1}{8}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$$

TOTAL 16 marks

2. (a) Find $\frac{dy}{dx}$ when

(i) $y = e^{\sqrt{x}} + cos^{-1}(2x)$ [3]

$$\frac{dy}{dx} = e^{\sqrt{x}} \times \underbrace{\frac{1}{2} x^{-1/2}} - \frac{2}{\sqrt{1-4x^2}}$$

✓ ✓ ✓

(ii) $y = \frac{\ln(x^2)}{sin^{-1}x}$ [3]

$$\frac{dy}{dx} = \frac{\sin^{-1}x \times \frac{2x}{x^2} - \ln(x^2) \times \frac{1}{\sqrt{1-x^2}}}{(\sin^{-1}x)^2}$$

✓ ✓ ✓

$$\frac{\frac{2}{x}\sin^{-1}x - \frac{\ln x^2}{\sqrt{1-x^2}}}{(\sin^{-1}x)^2}$$

(b) Find the gradient of the curve $3x^2 + 2xy + (lny)^2 = 16$ at the point (2, 1). [4]

differentiating implicitly

$$6x + 2x\frac{dy}{dx} + 2y + 2(\ln y)\frac{1}{y}\frac{dy}{dx} = 0$$

substitute $x = 2$ $y = 1$ FT

$$12 + 4\frac{dy}{dx} + 2 + 2\ln 1 \ \frac{1}{1}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-14}{4} = \frac{-7}{2}$$

(c) A curve is defined by the parametric equations

$$y = \frac{t}{2t+3} \text{ and } x = e^{-2t}$$

Find the gradient of the curve at the point for which $t = 0$. [5]

$$\frac{dy}{dt} = \frac{2t+3-2t}{(2t+3)^2} = \frac{3}{(2t+3)^2}$$ ✓

$$\frac{dx}{dt} = -2e^{-2t}$$ ✓

$$\frac{dy}{dx} = \frac{3}{(2t+3)^2} \cdot \frac{1}{-2e^{-2t}}$$ ✓

when $t=0$

$$\frac{dy}{dx} = \frac{3}{3^2} \cdot -\frac{1}{2}$$ ✓ FT

$$= -\frac{1}{6}$$ ✓

(d) Let $f(x, y) = (x^2 + y^2)^5 + \ln(xy)$, find $\dfrac{\partial^2 f}{\partial x \partial y}$ [2]

$$\frac{\partial f}{\partial y} = 5(x^2+y^2)^4 \, 2y + \frac{x}{xy}$$

$$= 10y(x^2+y^2)^4 + \frac{1}{y} \checkmark$$

$$\frac{\partial^2 f}{\partial x \partial y} = 40y(x^2+y^2)^3 (2x)$$

$$= 80xy(x^2+y^2)^3 \checkmark$$

TOTAL 17 marks

3. (a) (i) Express $f(x) = \frac{10x+9}{(2x+1)(2x+3)^2}$ in partial fractions. [5]

$$\frac{10x-9}{(2x+1)(2x+3)^2} = \frac{A}{2x+1} + \frac{B}{2x+3} + \frac{C}{(2x+3)^2} \checkmark$$

using a correct method to find A, B and C ✓

$$= \frac{1}{2x+1} \checkmark - \frac{1}{2x+3} \checkmark + \frac{3}{(2x+3)^2} \checkmark$$

(ii) Hence show that $\int_0^1 f(x)dx = \frac{1}{2}ln\left(\frac{9}{5}\right) + \frac{1}{5}$. [5]

$$= \int_0^1 \frac{1}{2x+1} - \frac{1}{2x+3} + \frac{3}{(2x+3)^2}\, dx$$

$$= \frac{1}{2}\ln(2x+1) \checkmark - \frac{1}{2}\ln(2x+3) \checkmark - \frac{3}{2(2x+3)} \checkmark \Big|_0^1$$

$$= \left[\frac{1}{2}\ln 3 + \frac{1}{2}\ln 5 - \frac{3}{2(5)}\right] - \left[\frac{1}{2}\ln 1 - \frac{1}{2}\ln 3 - \frac{1}{2}\right] \checkmark$$

$$= \frac{1}{2}(\ln 3 + \ln 3 - \ln 5 - \ln 1) + \frac{1}{2} - \frac{3}{10}$$

$$= \frac{1}{2}(\ln 9 - \ln 5) + \frac{1}{5}$$

$$= \frac{1}{2}\ln\left(\frac{9}{5}\right) + \frac{1}{5} \checkmark$$

(b) Using the substitution $u = x^2$, find

$\int_1^2 \frac{4x}{1+x^4}\,dx$ (give your answer to 2 decimal places) [5]

$u = x^2 \qquad \frac{du}{dx} = 2x \qquad dx = \frac{du}{2x}$

when $x = 1 \quad u = 1$ ✓

$x = 2 \quad u = 4$

substituting, integral becomes

$\int_1^4 \frac{2}{1+u^2}\,du$ ✓

$= 2\tan^{-1} u \Big|_1^4$ ✓

$= 2\tan^{-1} 4 - 2\tan^{-1} 1$ ✓

$= 2.652 - 1.571$ ✓ must use radian measure

$= 1.081 \simeq 1.08$

(c) It is given that for $n \geq 0$, $I_n = \int_0^{\frac{1}{2}} (1-2x)^n e^x \, dx$

(i) Show that for $n \geq 1$

$$I_n = 2nI_{(n-1)} - 1$$ [4]

Integrating by parts

$= (1-2x)^n \int_0^{1/2} e^x \, dx - \int_0^{1/2} e^x (-2n(1-2x)^{n-1} \, dx$

$= (1-2x)^n e^x \Big|_0^{1/2} + 2n \int_0^{1/2} e^x (1-2x)^{n-1} \, dx$

$= -1 + 2n I_{n-1}$

$= 2n I_{n-1} - 1$

(ii) Find the exact value of I_3. [4]

using reduction formula

$$I_3 = 2(3)I_2 - 1$$

$$= 6I_2 - 1$$

$$= 6[2(2)I_1 - 1] - 1$$

$$= 24I_1 - 7$$

$$= 24[2(1)I_0 - 1] - 7$$

$$= 48I_0 - 31$$

$$I_0 = \int_0^{1/2} e^x \, dx = e^{1/2} - 1$$

$$I_3 = 48(e^{1/2} - 1) - 31$$

$$= 48e^{1/2} - 79$$

(d) Use the trapezium rule with 4 trapezia of equal width to estimate the value of

$\int_0^4 \frac{1}{1+\sqrt{x}}\,dx$. Give your answer to 2 decimal places. [4]

TOTAL 27 marks

x	$\frac{1}{1+\sqrt{x}}$	
0	1	
1		0.5
2		0.4142
3		0.3660
4	1/3	
Σ	1.3333	1.2802

$$\int_0^4 \frac{1}{1-\sqrt{x}}\,dx = \frac{1}{2}\cdot 1\cdot\left(1.3333 + 2(1.2802)\right)$$

$$= 1.94685$$

$$\approx 1.95$$

END OF EXAMINATION