CARIBBEAN ADVANCED PROFICIENCY EXAMINATION

SCHOOL BASED ASSESSMENT

PURE MATHEMATICS

UNIT 1 – TEST 3

Time: 1 hour and 20 mimutes

This examination paper consists of 9 printed pages.

The paper consists of 6 questions.

The maximum mark for this examination is 60.

INSTRUCTIONS TO CANDIDATES

1. Write your answers in the spaces provided.
2. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided. You must also write your name and candidate number clearly on any additional paper used.
3. Answer **ALL** questions.
4. Unless otherwise stated in the question, any numerical answer that is not exact, **MUST** be written correct to three (3) significant figures.

EXAMINATION MATERIALS ALLOWED

1. Mathematical formulae
2. Electronic calculator (non-programmable, non-graphical).

1. a) Determine

$$\lim_{x \to -3} \frac{x^2+5x+6}{2x+6}$$ [3]

$$= \lim_{x \to -3} \frac{(x+3)(x+2)}{2(x+3)}$$ ✓ ✓

$$= \lim_{x \to -3} \frac{x+2}{2}$$

$$= \frac{-3+2}{2} = -\frac{1}{2}$$ ✓

b) Differentiate $f(x) = x^2 - 4x + 3$, using first principles. [6]

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ ✓ S.O.I.

$$f(x+h) = (x+h)^2 - 4(x+h) + 3$$
$$= x^2 + 2xh + h^2 - 4x - 4h + 3$$ ✓

$$f'(x) = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h + 3 - (x^2 - 4x + 3)}{h}$$ ✓

$$= \lim_{h \to 0} \frac{2xh + h^2 - 4h}{h}$$ ✓

$$= \lim_{h \to 0} 2x + h - 4$$ ✓

$$= 2x - 4$$ ✓

c) The function $f(x)$ is defined by $f(x) = \begin{cases} 2x+1 & x \le 3 \\ 10 - ax & x > 3 \end{cases}$

Find i) $\lim_{x \to 3^-} f(x)$ [1]

$= \lim_{x \to 3} (2x+1)$

$= 2(3) + 1 = 7$ ✓

ii) the value of a for which the function $f(x)$ is continuous [3]

$f(x)$ is continuous if $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$ ✓ S.O.I.

i.e $7 = 10 - a(3)$ ✓

$3a = 10 - 7 = 3$

$a = 1$ ✓

Total 13 marks

2. A curve C has equation

$$y = \frac{3}{(5-3x)^2}, x \neq \frac{5}{3}$$

The point P on C has x-coordinate 2. Find an equation of the normal to the curve C at P in the form $ax + by + c = 0$, where a, b and c are integers. [7]

Total 7 marks

when $x = 2$, $y = \frac{3}{(5-3(2))^2} = 3$ ✓

$\frac{dy}{dx} = \frac{18}{(5-3x)^3}$ ✓

gradient of tangent at $x = 2$

$= \frac{18}{(5-3(2))^3} = -18$ ✓ F.T.

gradient of normal at $x=2$

$m = \frac{1}{18}$ ✓

equation of normal : $y = \frac{1}{18}x + c$

at $(2, 3)$: $3 = \frac{1}{18}(2) + c$

$\Rightarrow c = \frac{26}{9}$ ✓ F.T.

equation of normal : $y = \frac{1}{18}x + \frac{26}{9}$ ✓

equation of normal in required form

$18y = x + 52$

$x - 18y + 52 = 0$ ✓

3. A curve has equation

$$y = \frac{x}{4+x^2}.$$

a) Use calculus to find the coordinates of the turning points of the curve. [5]

$\frac{dy}{dx} = \frac{4-x^2}{(4+x^2)^2}$ ✓

at turning points $\frac{dy}{dx} = 0$ ✓

i.e. $4 - x^2 = 0$

$x^2 = 4$ ✓

$x = \pm 2$

when $x = 2$ $y = \frac{2}{4+2^2} = \frac{1}{4}$ ✓

when $x = -2$ $y = \frac{-2}{4+(-2)^2} = -\frac{1}{4}$ ✓

so turning points are

$(2, \frac{1}{4})$ and $(-2, -\frac{1}{4})$

4.

b) Show that

$$\frac{d^2y}{dx^2} = \frac{2x(x^2 - 12)}{(4 + x^2)^3}$$

[4]

$$= \frac{2x(4+x^2)\left[-(4+x^2) - 2(4-x^2)\right]}{(4+x^2)^4}$$ ✓ ✓

$$= \frac{2x(-4-x^2-8+2x^2)}{(4+x^2)^3}$$ ✓

$$= \frac{2x(x^2-12)}{(4+x^2)^3}$$ ✓

c) Determine the nature of each of the turning points.

[4]

when $x=2$, $\frac{d^2y}{dx^2} = \frac{4(-8)}{8^3} < 0$ ✓

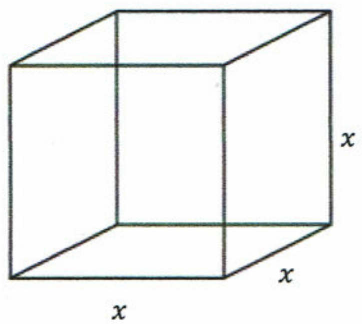
so $(2, \frac{1}{4})$ is a maximum point. ✓

when $x=-2$ $\frac{d^2y}{dx^2} = \frac{-4(-8)}{8^3} > 0$ ✓

so $(-2, -\frac{1}{4})$ is a minimum point ✓

Total 13 marks

4. The figure below shows a metal cube which is expanding uniformly as it is heated. At time t seconds, the length of each edge of the cube is $x\ cm$, and the volume of the cube is $V\ cm^3$.



a) Show that $\frac{dV}{dx} = 3x^2$ [1]

$V = x^3$

$\frac{dv}{dx} = 3x^{3-1} = 3x^2$ ✓

b) Given that the volume, $V\ cm^3$, increases at a constant rate of $0.048\ cm^3s^{-1}$, find

i) $\frac{dx}{dt}$, when $x = 8$ [3]

$\frac{dv}{dt} = 0.048\ cm^3 s^{-1}$ ✓

$\frac{dx}{dt} = \frac{dv}{dt} \times \frac{1}{\frac{dv}{dx}}$

$= 0.048 \times \frac{1}{3x^2}$ ✓

when $x = 8$

$\frac{dx}{dt} = 0.048 \times \frac{1}{3(8^2)}$

$= 0.00025\ cm\ s^{-1}$ ✓

6.

ii) The rate of increase of the total surface area, A, of the cube, in cm^2s^{-1}, when $x = 8$. [4]

$A = 6x^2$ ✓ $\frac{dA}{dx} = 12x$ ✓

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$= 12x \times 0.00025$ ✓

when $x = 8$

$\frac{dA}{dt} = 12(8) \times 0.00025$

$= 0.024 \text{ cm}^2 s^{-1}$ ✓

Total 8 marks

5. a) Use calculus to find the EXACT value of

$$\int_1^2 \left(3x^2 + 5 + \frac{4}{x^2}\right) dx$$

✓ ✓ ✓

$= \left[\frac{3x^3}{3} + 5x - \frac{4}{x}\right]_1^2$

$= (8 + 10 - 2) - (1 + 5 - 4)$ ✓

$= 16 - 2 = 14$ ✓

[5]

b) Find the value of k if $\int_2^k 6(1-x)^2 dx = 52$.

$$\int_2^K 6(1-x)^2 dx = \left[\frac{6(1-x)^3}{3(-1)}\right]_2^K = 52$$

$$= \left[-2(1-x)^3\right]_2^K = -2(1-K)^3 - \left[-2(1-2)^3\right] = 52 \quad \text{F.T.}$$

$$= -2(1-K)^3 - 2 = 52$$

$$(1-K)^3 = -27$$

$$1-K = \sqrt[3]{-27}$$

$$1-K = -3$$

$$K = 4$$

[5]

Total 10 marks

6. a) By using the substitution $U = sinx + cosx$, show that

$$\int (sinx - cosx)\,(sinx + cosx)^5 dx = -\frac{1}{6}(sinx + cosx)^6 + c$$

$u = \sin x + \cos x$

$\frac{du}{dx} = \cos x - \sin x \Rightarrow dx = \frac{du}{\cos x - \sin x}$

substituting

$$\int (\sin x - \cos x)\, u^5 \frac{du}{\cos x - \sin x}$$

$$= \int -(\cos x - \sin x)\, u^5 \frac{du}{\cos x - \sin x}$$

$$= -\int u^5 du = -\frac{u^6}{6} + c$$

$$= -\frac{(\sin x + \cos x)^6}{6} + c$$

[5]

8.

b) Hence, find the EXACT value of

$$\int_0^{\frac{\pi}{4}} (sinx - cosx)\,(sinx + cosx)^5 dx$$

when $x = \frac{\pi}{4}$ $\quad u = 2\frac{\sqrt{2}}{2} = \sqrt{2}$ ✓

when $x = 0$ $\quad u = 0 + 1 = 1$ ✓

$$\int_0^{\pi/4} (\sin x - \cos x)\,(\sin x + \cos x)^5\,dx$$

$$= \int_0^{\sqrt{2}} -(u^5)\,du = \left[-\frac{u^6}{6}\right]_1^{\sqrt{2}}$$ ✓

$$= -\frac{(\sqrt{2})^6}{6} - \left(-\frac{1}{6}\right)$$

[4]

Total 9 marks

$$= -\frac{8}{6} + \frac{1}{6}$$

$$= -\frac{7}{6}$$ ✓ C.A.O.