CAPE – 2015

Pure Math Unit 2 – Paper 02

Solutions

Qu	Details	CK	AK	R	Tot
1 (a) (i)	$\frac{-2+2\mathrm{i}}{\sqrt{3}+3\mathrm{i}} \times \frac{\sqrt{3}-3\mathrm{i}}{\sqrt{3}-3\mathrm{i}} = \left(\frac{3-\sqrt{3}}{6}\right) + \left(\frac{3+\sqrt{3}}{6}\right)\mathrm{i}$				
(ii)	$r = \lvert w \rvert = \left\lvert \frac{-2+2\mathrm{i}}{1 \times \sqrt{3}+3\mathrm{i}} \right\rvert = \frac{\sqrt{6}}{3}$ $\arg w = \frac{3\pi}{4} - \left[\frac{\pi}{12} + \frac{\pi}{3}\right] = \frac{\pi}{3} \quad w = \frac{\sqrt{6}}{3}\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}$				
(b)	$(x+\mathrm{i}y)^2 = 2+\mathrm{i} \Rightarrow x^2 - y^2 = 2$ and $2xy = 1$ solving for x and y gives $4x^4 - 8x^2 - 1 = 0 \Rightarrow x^2 = \frac{2 \pm \sqrt{5}}{2}$ but $x^2 - y^2 > 0 \Rightarrow x^2 = \frac{2+\sqrt{5}}{2}$				
(c) (i)	$\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\sqrt{(1-t^2)}\left(-\mathrm{e}^{-t}\right) - \mathrm{e}^{-t}\left[\frac{1}{2}(1-t^2)^{-\frac{1}{2}}(-2t)\right]}{\left(\sqrt{1-t^2}\right)^2} = \frac{\mathrm{e}^{-t}\left(t^2+t-1\right)}{\sqrt{(1-t^2)^3}}$ $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\sqrt{1-t^2}} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-t^2}} \times \frac{\sqrt{(1-t^2)^3}}{\mathrm{e}^{-t}\left(t^2+t-1\right)} = \frac{\mathrm{e}^{t}\left(1-t^2\right)}{\left(t^2+t-1\right)}$				
(ii)	$1 - t^2 = 0 \Rightarrow t = \pm 1$ but x is undefined for ± 1				

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2 (a) (i)	$8x + 3y^2 + 6xy\frac{\mathrm{d}y}{\mathrm{d}x} + 7 + 3\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{8x + 3y^2 + 7}{3(1 + 2xy)}$ Award full marks for correct working				
(ii)	$\frac{\partial \mathrm{f}(x, y)}{\partial y} = 6xy + 3$ $\frac{\partial^2 \mathrm{f}(x, y)}{\partial y^2} = 6x$ $\frac{\partial \mathrm{f}(x, y)}{\partial x} = 8x + 3y^2 + 7$ $\frac{\partial \mathrm{f}(x, y)}{\partial y\, \partial x} = 6y$ $6(6xy + 3) - 10 = 6x(6y) + 8 \Rightarrow$ result shown				
(b) (i)	$\frac{18x^2 + 13}{9x^2 + 4} = 2 + \frac{5}{9x^2 + 4}$				
(ii)	$2\int_0^2 \left(2 + \frac{5}{4\left[1 + \left(\frac{3x}{2}\right)^2\right]} \right) \mathrm{d}x = 2\left(2x + \frac{5}{6}\tan^{-1}\left(\frac{3x}{2}\right)_0^2 \right)$ $8 + \frac{5}{3}\tan^{-1}(3)$				
(c) (i)	$\int (h^n \ln h)\, \mathrm{d}h = h^n h(\ln h - 1) - \int n h^{n-1} h(\ln h - 1)\, \mathrm{d}h$ $= h^{n+1}(\ln h - 1) - n\int h^n \ln h\, \mathrm{d}h + n\int h^n\, \mathrm{d}h$ $(n + 1)\int h^n \ln h\, \mathrm{d}h = h^{n+1}\ln h - h^{n+1} + \frac{n}{n+1}h^{n+1}$ $\int h^n \ln h\, \mathrm{d}h = \frac{h^{n+1}}{n+1}\left(\ln h - 1 + \frac{n}{n+1}\right)$ $= \frac{h^{n+1}}{(n+1)^2}\left[(n+1)\ln h - 1\right] + C$				

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2 (c) (ii)	$\int\left(\sin^2 x\cos x\ln(\sin x)\right)\mathrm{d}x=\frac{\sin^{(2+1)}x}{(2+1)^2}\left[(2+1)\ln(\sin x)-1\right]+C$ $\frac{\sin^3 x}{9}\left[3\ln(\sin x)-1\right]+C$				
3 (a) (i)	$\frac{\lim_{x\to\infty} 2n+1}{\lim_{x\to\infty} n\sqrt{\left(1+\frac{1}{n^2}\right)}}$ $=\frac{\lim_{x\to\infty} 2n}{\lim_{x\to\infty} n\sqrt{\left(1+\frac{1}{n^2}\right)}}+\frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty} n\sqrt{\left(1+\frac{1}{n^2}\right)}}$ $=\lim_{x\to\infty}\frac{2}{\sqrt{\left(1+\frac{1}{n^2}\right)}}=2$				
(ii)	$T_4=\frac{2(4)+1}{\sqrt{4^2+1}}=\frac{9}{\sqrt{17}}=\frac{9}{4\sqrt{\left(1+\frac{1}{16}\right)}}=\frac{9}{4}\left(1+\frac{1}{16}\right)^{-\frac{1}{2}}$				
(iii)	$\frac{9}{4}\left[1+\left(-\frac{1}{2}\right)\left(\frac{1}{16}\right)+\frac{\left(-\frac{1}{2}-1\right)}{2}\left(\frac{1}{16}\right)^2+\frac{\left(-\frac{1}{2}-2\right)}{6}\left(\frac{1}{16}\right)^3\right]$ ≈ 2.18				
(b) (i)	$S_n=\sum_{r=1}^{n}\frac{r+1}{r^2}$				

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(b) (ii)	$S_n = \sum_{r=1}^{n} \frac{1}{r} + \frac{\pi^2}{6} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \frac{\pi^2}{6}$ $1 + \frac{1}{2} < 1 + \frac{1}{2} + \frac{1}{3} < 1 + \frac{1}{2} + \frac{1}{3} < \frac{1}{4} < \ldots$ $\Rightarrow S_n$ diverges as $n \to \infty, \forall\, n \in N$ Knowledge of the harmonic series not required				
(c)	The statement $P_n = \sum_{r=1}^{n} r(r-1) = \frac{n(n^2-1)}{3}$ for $n = 1$ P_1 is true for $n = 2$ P_2 is true assume P_n is true *for* $n = k,\ k \in N,\ k > 2$, then $P_k = \frac{k(k^2-1)}{3}$ $P_{k+1} = \frac{k(k^2-1)}{3} + (k+1)(k+1-1) = \frac{k}{3}(k+1)(k-1) + (k+1)$ $= \frac{k+1}{3}(k(k-1)+3k) = \frac{k+1}{3}\left[(k+1)^2 - 1\right]$ since P_k is true and P_{k+1} is true $\Rightarrow P_n$ is true $\forall\, n \in N$				
4 (a) (i)	$g(0) = e \quad g'(x) = 3e^{3x+1} \quad g'(0) = 3e \quad g''(x) = 9e^{3x+1} \quad g''(0) = 9e$ $g'''(x) = 27e^{3x+1} \quad g'''(0) = 27e \quad g''''(x) = 81e^{3x+1} \quad g''''(0) = 81e$ $g(x) = e + 3ex + \frac{9ex^2}{2} + \frac{9ex^3}{2} + \frac{27ex^4}{8}$				
(ii)	$g(0.2) = e + 3e(0.2) + \frac{9e(0.2)^2}{2} + \frac{9e(0.2)^3}{2} + \frac{27e(0.2)^4}{8}$ $= 4.951\ldots$				
(b) (i)	$f(-2) < 0 \quad f(-1) > 0 \quad$ or $f(-1) > 0 \quad f(0) < 0$ Alternatively consider $\frac{-2+0}{2} = -1 \Rightarrow$ IVT may be applied to $(-2, -1)$ and $(-1, 0)$				

Qu	Details	CK	AK	R	Tot
4 (b) (ii)	$\frac{-0.7-0.3}{2}=-0.5 \quad f(-0.5)<0 \quad \frac{-0.7-0.5}{2}=-0.6$ $f(-0.6)>0 \quad \frac{-0.6-0.5}{2}=-0.55 \quad f(-0.55)>0$ $\frac{-0.55-0.5}{2}=-0.525 \quad f(-0.525)<0$ $\frac{-0.55-0.525}{2}=-0.5375 \quad f(-0.5375)\approx-0.001.....$ $\Rightarrow f(-0.538)\approx 0$				
(c)	$x_2=5.5-\left[\frac{\sin(16.5)}{3\cos(16.5)}\right]=5.16218...$ $x_3=5.16218-\left[\frac{\sin(15.48654)}{3\cos(15.48654)}\right]=5.23721......$ $x_4=5.23721-\left[\frac{\sin(15.71163)}{3\cos(15.71163)}\right]=5.23598$ $x\approx 5.24...$				
5 (a) (i)	${}^{10}C_4=210$				
(ii)	${}^{3}C_1\times{}^{7}C_3+{}^{3}C_2\times{}^{7}C_2+{}^{3}C_3\times{}^{7}C_1=175$				
(b) (i)	Single numbers 1, 2, 3, 4, 5 = 5 2-digit numbers = ${}^{5}P_2=20$ 3-digit numbers = ${}^{5}P_3=60$ 4-digit numbers = ${}^{5}P_4=120$ 5-digit numbers = ${}^{5}P_5=120$ total numbers = 325				
(ii)	3-digit numbers > 100, > 200, > 300, > 400, > 500 = $5\times{}^{4}P_2=60$ 4-digit numbers > 1000, > 2000 > 3000, > 4000, > 5000 = $5\times{}^{4}P_3=120$ 5-digit numbers > 10 000, > 20 000, > 30 000, > 40 000, > 50 000 = $5\times{}^{4}P_4=120$ P (#s > 100) = $\frac{300}{325}=\frac{12}{13}$				

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<tr><td>5 (c) (i)</td><td>

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & -\frac{7}{2} \\ 1 & -1 & 2 & 7 \\ \frac{3}{2} & 0 & 3 & 9 \end{array}\right) \quad 2R_1 + 6R_2 \,\&\, 2R_3 \Rightarrow \left(\begin{array}{ccc|c} 4 & 6 & -2 & 7 \\ 10 & 0 & 10 & 35 \\ 3 & 0 & 6 & 18 \end{array}\right)$$

$$3R_2 - 10R_3 \Rightarrow \left(\begin{array}{ccc|c} 4 & 6 & -2 & -7 \\ 10 & 0 & 10 & 35 \\ 0 & 0 & -30 & -75 \end{array}\right)$$

$$5R_1 - 2R_2 \Rightarrow \left(\begin{array}{ccc|c} 4 & 6 & -2 & -7 \\ 0 & 30 & -30 & -105 \\ 0 & 0 & -30 & -75 \end{array}\right)$$

$$\frac{1}{4}R_1 \,\&\, \frac{1}{30}R_2 \,\&\, -\frac{1}{30}R_3 \Rightarrow \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & -\frac{7}{4} \\ 0 & 1 & -1 & -\frac{7}{2} \\ 0 & 0 & 1 & \frac{5}{2} \end{array}\right)$$

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$x = 1, \quad y = -1, \quad z = \frac{5}{2}$

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$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & -\frac{7}{2} \\ 1 & -1 & 2 & 7 \\ \frac{3}{2} & -\frac{3}{2} & 3 & 9 \end{array}\right) \quad R_2 - \frac{2}{3}R_3 \Rightarrow \left(\begin{array}{ccc|c} 2 & 3 & -1 & -\frac{7}{2} \\ 1 & -1 & 2 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Since the last row is inconsistent no solution exists
Note: showing the determinant = 0 is not sufficient for no solution exists.....the system may have infinitely many solutions

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Qu	Details	CK	AK	R	Tot
6 (a) (i)	rain = 0.32 → school = 0.7 → r & s = 0.224 rain = 0.32 → no school = 0.3 → r & no school = 0.096 sunny = 0.68 → school = 0.99 → s & school = 0.6732 sunny = 0.68 → no school = 0.01 → s & no school = 0.0068				
(ii)	P (school) = 0.224 + 0.6732 = 0.8972				
(iii)	P (rain given school) = $\frac{0.224}{0.8972} = 0.250...$				
(b)	$\frac{dy}{dx} - \left(\frac{1}{x(1+x)}\right)y = -\frac{x}{1+x}$ $I = e^{\int -\frac{1}{x(1+x)}dx} = e^{\int\left(\frac{1}{(1+x)} - \frac{1}{x}\right)dx} = e^{\ln\left(\frac{1+x}{x}\right)} = \frac{1+x}{x}$ $\left(\frac{1+x}{x}\right)\frac{dy}{dx} - \left(\frac{1}{x(1+x)}\right)\left(\frac{1+x}{x}\right)y = \left(-\frac{x}{1+x}\right)\left(\frac{1+x}{x}\right)$ $\int \frac{d}{dx}\left(\frac{(1+x)}{x}y\right)dy = \int - dx$ $\frac{(1+x)}{x}y = -x \Rightarrow y + xy + x^2 = 0$				

Qu	Details	CK	AK	R	Tot
6 (b) (ii) a)	$u^2 = 2 \Rightarrow u = \pm\sqrt{2}$ CF: $y = Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x}$ $y(0) = 1 \Rightarrow A + B = 1$ ....(i) $y' = \sqrt{2}Ae^{\sqrt{2}x} - \sqrt{2}Be^{-\sqrt{2}x}$ $y'\left(\frac{\sqrt{2}}{2}\right) = 0 \Rightarrow 0 = \sqrt{2}Ae - \sqrt{2}Be^{-1}$ ....(ii) $\sqrt{2}e^{-1}(i) \Rightarrow \sqrt{2}Ae^{-1} + \sqrt{2}Be^{-1} = \sqrt{2}e^{-1}$ and $\sqrt{2}Ae - \sqrt{2}Be^{-1} = 0$ $(i) + (ii) \Rightarrow \sqrt{2}\,e^{-1} = \sqrt{2}Ae + \sqrt{2}Ae^{-1} \Rightarrow \sqrt{2}\,e^{-1} = \sqrt{2}A\left(e + e^{-1}\right)$ $A = \dfrac{\sqrt{2}\,e^{-1}}{\sqrt{2}\left(\dfrac{e^2+1}{e}\right)}$ $A = \dfrac{1}{e^2+1}$ $B = 1 - A = 1 - \dfrac{1}{e^2+1} = \dfrac{e^2}{e^2+1}$ $y = \dfrac{1}{e^2+1}\left(e^{\sqrt{2}x} + e^{2-\sqrt{2}x}\right)$				