CAPE – 2015

Pure Math Unit 1 – Paper 02

Solutions

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(ii)

(iii)</td><td>

$\sim p \to \sim q$ **$\sim q \to \sim p$**

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<tr><th>p</th><th>q</th><th>~ p</th><th>~ q</th><th>p → q</th><th>~ q → ~ p</th></tr>
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Logically equivalent - truth values the same

Note:

$p \to q = \sim p \vee q$ and **$\sim q \to \sim p = \sim\sim q \vee \sim p = q \vee \sim p = \sim p \vee q$**

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<tr><td>(b) (i)

(ii)</td><td>

$f(5) = 0 \Rightarrow 25p + q = -120$ and $f(1) = 24 \Rightarrow p + q = 24$

$p = -6 \quad q = 30$

$f(x) = (x-5)(x+2)(x-3)$

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<tr><td>(c)</td><td>

$P_n \Rightarrow 4(S_n) = 5^{n+1} - 5 \quad P_1$ is true $\; P_2$ is true

Assume true for $P_k \Rightarrow 4(S_k) = 5^{k+1} - 5 \quad k \in \mathbb{N}$

Then $P_{k+1} \Rightarrow 4(S_{k+1}) = (5^{k+1} - 5) + 4(5^{k+1}) = 4\times 5^{k+1} + 5^{k+1} - 5$

$= 5\times 5^{k+1} - 5 = 5^{(k+1)+1} - 5$

Since true for P_k *and* P_{k+1} then true $\forall\, P_n \quad n \in \mathbb{N}$

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Qu	Details	CK	AK	R	Tot
2 (a) (i)	let $(g \circ f)(a) = (g \circ f)(a')$ i.e. $gf(a) = gf(a')$ then $f(a) = f(a')$ since g is one-to-one and $a = a'$ since f is one-to-one then $(g \circ f)$ is one-to-one				
(ii)	Let $c \in C$ since g is onto there exists $b \in B$ such that $g(b) = c$ since f is onto there exists $a \in A$ such that $f(a) = b$ $(g \circ f)(a) = gf(a) = g(b) = c$ hence each $a \in C$ is the image of some element $a \in A$ then $(g \circ f)$ is an onto function				
(b) (i)	$3 - 4\left(3^{-2x}\right) - 4\left(3^{-2x}\right)^2 = 0 \quad y = 3^{-2x} = 4y^2 + 4y - 3 = 0$				
(ii)	$y = 3^{-2x} = \frac{1}{2} \Rightarrow x = -\frac{1}{2}\frac{\ln\left(\frac{1}{2}\right)}{\ln(3)} = 0.31546...$ $y = 3^{-2x} = -\frac{3}{2}$ (no solution) $5x - 6 = x + 5 = x = \frac{11}{4} \quad 6 - 5x = x + 5 \Rightarrow x = \frac{1}{6}$				
(c) (i)	$N = 301$				
(ii)	$903 = 300 + 5^t \Rightarrow t = \frac{\ln(903 - 300)}{\ln 5} = 3.9777... \approx 4$ hrs				

Qu	Details	CK	AK	R	Tot
3 (a) (i)	$\cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$ $= \left(2\cos^2 x - 1\right)\cos x - \left(2\sin x\cos x\right)\sin x$ $= 2\cos^3 x - \cos x - 2\left(1 - \cos^2 x\right)\cos x$ $= 4\cos^3 x - 3\cos x$				
(ii)	$-2\sin 4x \sin 2x = 0 \Rightarrow \sin 4x = 0 \quad \sin 2x = 0$ $x = \dfrac{n\pi}{4} \quad x = \dfrac{n\pi}{2}$ $x = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi, \dfrac{5\pi}{4}, \dfrac{3\pi}{2}, \dfrac{7\pi}{4}, 2\pi$ Alternative: from (i) $4\cos^3 2x - 3\cos 2x = 0 \Rightarrow 4\cos 2x\left(\cos^2 2x - 1\right) = 0$ $2x = (2n+1)\dfrac{\pi}{2}, \quad 2x = \pm 2n\pi$ $x = 0, \dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \pi, \dfrac{5\pi}{4}, \dfrac{3\pi}{2}, \dfrac{7\pi}{4}, 2\pi$				
(b) (i) (ii)	$f(2\theta) = 5\sin\left(2\theta + 0.295\pi\right)$ $2 \le 7 - f(2\theta) \le 12 \qquad \dfrac{1}{12} \le \dfrac{1}{7 - f(2\theta)} \le \dfrac{1}{2}$				
4 (a) (i)	$C_1: \ (x+3)^2 + (y-2)^2 = 10 \quad C_2: \ (x-3)^2 + (y-2)^2 = 16$ $12x + 6 = 0 \quad x = -\dfrac{1}{2}$ $y = \dfrac{4 \pm \sqrt{15}}{2}$				
(b)	$A = (0, 3) \ \ B(5, 2) \quad P(x, y) \quad (PA)^2 = 2(PB)^2$ $(x-0)^2 + (y-3)^2 = 2^2\left[(x-5)^2 + (y-2)^2\right]$ $3x^2 + 3y^2 - 40x - 10y + 107 = 0$ $\left(x - \dfrac{20}{3}\right)^2 + \left(y - \dfrac{5}{3}\right)^2 = \dfrac{104}{9}$				
5 (a)	$\lim\limits_{x\to 0} \dfrac{a\sin(ax)}{ax} = a \quad \lim\limits_{x\to 0} a = 4 \Rightarrow a = 4$				

Qu	Details	CK	AK	R	Tot
5 (b)	$f'(x) = \lim_{h \to 0} \frac{\sin(2(x+h)) - \sin(2x)}{h} = \lim_{h \to 0} \frac{2\cos(2x+h)\sin(h)}{h}$ $= \lim_{h \to 0} 2\cos(2x+h) \times \lim_{h \to 0} \frac{\sin(h)}{h} = 2\cos(2x)$				
(c) (i) (ii)	$\frac{dy}{dx} = \frac{\sqrt{(1+x^2)}(2) - 2x\left[\frac{1}{2} \times 2x(1+x^2)^{-\frac{1}{2}}\right]}{1+x^2} = \frac{2(1+x^2) - 2x^2}{(1+x^2)\sqrt{(1+x^2)}}$ $\frac{dy}{dx} = \frac{1}{1+x^2} \times \frac{2}{\sqrt{(1+x^2)}} \Rightarrow x\frac{dy}{dx} = \frac{y}{1+x^2}$ $\frac{dy}{dx} = \frac{2}{(1+x^2)^{\frac{3}{2}}} \Rightarrow \frac{d^2y}{dx^2} \frac{(1+x^2)^{\frac{3}{2}}(0) - 2\left[\frac{3}{2} \times 2x(1+x^2)^{\frac{1}{2}}\right]}{(1+x^2)^3}$ $\frac{d^2y}{dx^2} \frac{-6x}{(1+x^2)^2\sqrt{(1+x^2)}} = \frac{-3y}{(1+x^2)^2}$ $\frac{-3y}{(1+x^2)^2} + \frac{3y}{(1+x^2)^2} = 0$				
6 (a) (i) (ii)	A: $3x - 7 = 9 - x$ $\quad x = 4, y = 5$ B: $3x - 7 = \frac{x+3}{3}$ $\quad x = 3, y = 2$ C: $\frac{x+3}{3} = 9 - x$ $\quad x = 6, y = 3$ Area = $\int_3^4 (3x-7)\,dx + \int_4^6 (9-x)\,dx - \int_3^6 \left(\frac{x+3}{3}\right) dx = 4 \text{ units}^2$				
(b) (i)	$\int_{-6}^{y} dy = \int_0^x (3x^2 + 8x - 3)\,dx \Rightarrow y = x^3 + 4x^2 - 3x - 6$				

Qu	Details	CK	AK	R	Tot
(b) (ii)	$3x^2 + 8x - 3 = 0 \quad \left(\frac{1}{3}, -\frac{176}{27}\right) \quad (-3, 12)$ $\left(\frac{d^2y}{dx^2}\right)_{\frac{1}{3}} > 0 \Rightarrow \left(\frac{1}{3}, -\frac{176}{27}\right)_{\min} \quad \left(\frac{d^2y}{dx^2}\right)_{-3} < 0 \Rightarrow (-3, 12)_{\max}$				
(iii)