HARRISON COLLEGE INTERNAL EXAMINATION MARCH 2014
CARIBBEAN ADVANCED PROFICIENCY EXAMINATION
SCHOOL BASED ASSESSMENT
PURE MATHEMATICS
PREVIEW UNIT 1 – TEST 1
1 hour 30 minutes

This examination paper consists of **2** printed pages.
This paper consists of **9** questions.
The maximum mark for this examination is **60**.

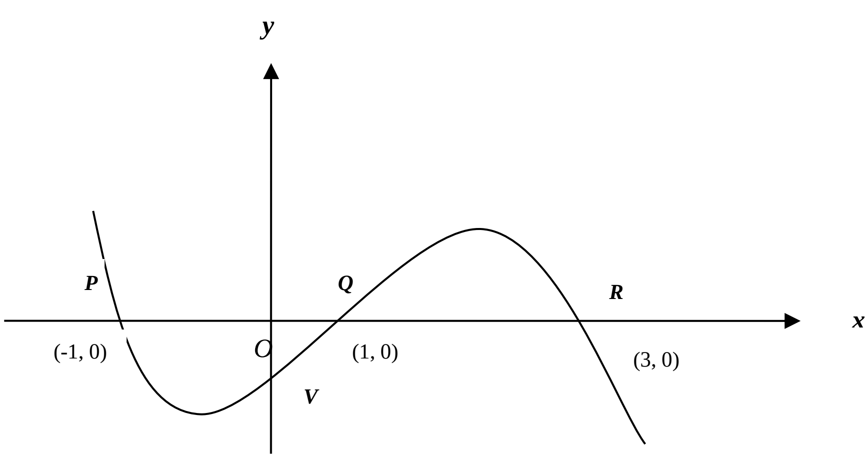
INSTRUCTIONS TO CANDIDATES
(i) Write your name clearly on each sheet of paper used
(ii) Answer **ALL** questions
(iii) Number your questions identically as they appear on the question paper and do **NOT write your solutions to different questions** beside each other
(iv) Unless otherwise stated in the question, any numerical answer that is not exact, **MUST** be written correct to three (3) significant figures

EXAMINATION MATERIALS ALLOWED
(a) Mathematical formulae
(b) Scientific calculator (non-programmable, non-graphical)

1) Given that $\boldsymbol{p}$ and $\boldsymbol{q}$ are propositions, use the algebra of propositions to show that
$(\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \wedge \boldsymbol{q}) \equiv (\boldsymbol{p} \wedge \boldsymbol{q})$ [3]

2) (i) Evaluate $\sum_{r=5}^{500}(2r+5)$. [4]
(ii) Given that $\sum_{r=1}^{n}(pr+q) = 2n^2 + n$, find the constants p and q. [4]

3) (a) The sketch below, **not drawn to scale**, shows part of the graph of $y = -x^3 + bx^2 + cx + d$, where b, c and d are constants.



The points P, Q and R have coordinates (–1, 0), (1, 0) and (3, 0) respectively.

The curve crosses the y-axis at V.

(i) Evaluate b, c and d. [6]

(ii) Determine the coordinates of V. [1]

(b) Solve the equation $2x^3 + x = 3$ for $x \in \mathbf{R}$. [6]

4) Prove by mathematical induction that $\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)} = \frac{n}{4n+1} \ \forall\, n \in N$. [6]

5) (a) By using the substitution $u = 3^x$, solve the equation $9^x - 5(3^x) + 4 = 0$. [4]

(b) Solve for x the equation $e^{3x} = 4 - 4e^{-3x}$, giving your answer in terms of ln. [4]

6) The rate of growth of a bacterial colony is given by $P(t) = 2 + 5^t$, where $P(t)$ represents the number of bacteria at time t minutes.

Determine

(i) the initial number of bacteria in the colony [2]

(ii) the length of time taken for the number of bacteria in the colony to reach 1 000 000. [4]

7) The function f is defined by $f: x \rightarrow \ln 2x: x \in \mathbf{R}, x > 0$.

(i) Sketch the graph of f, showing clearly any intersection with the axes. [2]

(ii) Determine an expression for the inverse function, $f^{-1}(x)$. [3]

(iii) State the domain, and the range of $f^{-1}(x)$. [2]

The function g is defined by $g: x \rightarrow \frac{1}{2}x - 4, x \in \mathbf{R}$.

(iv) Determine $fg(x)$. [2]

8) Find the range of values of $x \in \mathbf{R}$ for which $\frac{3-x}{2x-5} \geq 0, x \neq \frac{5}{2}$ [3]

9) Solve for $x \in \mathbf{R}$, $x + 4 = |2x|$ [4]

End of Preview Test

PREVIEW SOLUTIONS – CAPE 2014: UNIT 1 TEST 1

1) $(\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \wedge \boldsymbol{q}) \equiv (\boldsymbol{p} \wedge \boldsymbol{q})$

Proof: LHS

$(\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \wedge \boldsymbol{q})$

$= [(\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{p}] \wedge [(\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{q}]$ distributive [1 + 1 marks]

$= [\boldsymbol{p}] \wedge [\boldsymbol{q}]$ absorption [1 mark]

$= (\boldsymbol{p} \wedge \boldsymbol{q})$

2) (i) $\sum_{r=5}^{500}(2r+5)$

$= \sum_{r=1}^{500}(2r+5) - \sum_{r=1}^{4}(2r+5)$ [1 mark]

$= [2\sum_{r=1}^{500} r + \sum_{r=1}^{500} 5] - [2\sum_{r=1}^{4} r + \sum_{r=1}^{4} 5]$ [1 mark]

$= 2[\frac{1}{2}(500)(500+1)] + [(5 \times 500)] - (2[\frac{1}{2}(4)(4+1)] + [(5 \times 4)])$ [1 mark]

$= 252\ 960$ [1 mark]

(ii) $\sum_{r=1}^{n}(pr+q) = 2n^2 + n$

$p\left[\frac{1}{2}(n)(n+1)\right] + (q \times n) = 2n^2 + n$ [1 mark]

$p[(n)(n+1)] + 2qn = 4n^2 + 2n$

$pn^2 + pn + 2qn = 4n^2 + 2n$

$pn^2 + (p + 2q)n = 4n^2 + 2n$ [1 mark]

$p = 4$, $(p + 2q) = 2$ [1 mark]

$4 + 2q = 2$

$q = -1$ [1 mark]

3) (a) (i) $y = -x^3 + bx^2 + cx + d$ OR $(x+1)(x-1)(x-3) = 0$

$P(-1, 0)$; $Q(1, 0)$; $R(3, 0)$

$0 = 1 + b - c + d$ i.e. $b - c + d = -1$ Eqn 1 [1 mark]

$0 = -1 + b + c + d$ i.e. $b + c + d = 1$ Eqn 2 [1 mark]

$-2c = -2$

$c = 1$

$b + d = 0$ i.e. $d = -b$ [1 mark]

$0 = -27 + 9b + 3c + d$

$0 = -27 + 9b + 3 - b$ Eqn 3 [1 mark]

$b = 3$ [1 mark]

$d = -3$ [1 mark]

(ii) $V(0, d)$ so $V(0, -3)$ [1 mark]

(b) $2x^3 + x = 3$

$2x^3 + x - 3 = 0$ [1 mark]

$(x-1)(2x^2 + 2x + 3) = 0$ [1+1+1marks]

$x = 1$, no real solutions [1+1 marks]

4) Let P_n be the statement "$\sum_{r=1}^{n}\frac{1}{(4r-3)(4r+1)} = \frac{n}{4n+1}\ \forall\ n \in N$."

Basic Step – To Prove P_n is true for $n = 1$ i.e. To Prove P_1 is true

When $r = 1$, LHS $= \frac{1}{(1)(4+1)} = \frac{1}{5}$ $\Big\}$

When $n = 1$, RHS = $\frac{1}{4(1)+1} = \frac{1}{5}$ [1 mark]

$\therefore$ P_1 is true

Inductive Step – Assume P_n is true for $n = k$ i.e. Assume P_k is true

$$P_k \equiv \sum_{r=1}^{k} \frac{1}{(4r-3)(4r+1)}$$

$$= \frac{k}{4k+1} \ \forall\, k \in Z^+$$ [1 mark]

We are required to show that if P_k is true then

$$P_{k+1} \equiv \sum_{r=1}^{k+1} \frac{1}{(4r-3)(4r+1)}$$

$$= \frac{(k+1)}{4(k+1)+1}$$

$$= \frac{k+1}{4k+5} \ \forall\, k \in N$$

Now P_{k+1} = Sum of first k terms + $(k+1)^{\text{st}}$ term

$$= P_k + \frac{1}{[4(k+1)-3][4(k+1)+1]}$$

$$= \sum_{r=1}^{k} \frac{1}{(4r-3)(4r+1)} + \frac{1}{[4k+1][4k+5]}$$ [1 mark]

$$= \frac{k}{4k+1} + \frac{1}{[4k+1][4k+5]}$$

$$= \frac{k[4k+5]+1}{[4k+1][4k+5]}$$ [1 mark]

$$= \frac{4k^2+5k+1}{[4k+1][4k+5]}$$

$$= \frac{(4k+1)(k+1)}{[4k+1][4k+5]}$$

$$= \frac{k+1}{4k+5}$$ as required [1 mark]

$\therefore$ $P_k \Rightarrow P_{k+1}$ i.e. $P_1 \Rightarrow P_2$, $P_2 \Rightarrow P_3$ etc.

Hence, by MI, $\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)} = \frac{n}{4n+1} \ \forall\, n \in N$. [1 mark]

5) (a) $9^x - 5(3^x) + 4 = 0$

$u^2 - 5u + 4 = 0$ [1 mark]

$(u-4)(u-1) = 0$ [1 mark]

$u = 4$, $u = 1$

$3^x = 4$, $3^x = 1$

$x = \frac{\ln 4}{\ln 3}$, $x = 0$ [1 + 1 marks]

(b) $e^{3x} = 4 - 4e^{-3x}$

$(e^{3x})^2 = 4e^{3x} - 4$ [1 mark]

$(e^{3x})^2 - 4e^{3x} + 4 = 0$

$(e^{3x} - 2)(e^{2x} - 2) = 0$ [1 mark]

$e^{3x} = 2$ (twice)

$\ln (e^{3x}) = \ln 2$ [1 mark]

$3x\ln e = \ln 2$

$x = \frac{1}{3}\ln 2$ [1 mark]

6) (i) $P(t) = 2 + 5^t$

When $t = 0$; $P(0) = 2 + 5^0$ [1 mark]

= 3 bacteria [1 mark]

(ii) $P(t) = 1\,000\,000$

$1\,000\,000 = 2 + 5^t$

$999\,998 = 5^t$ [1 mark]

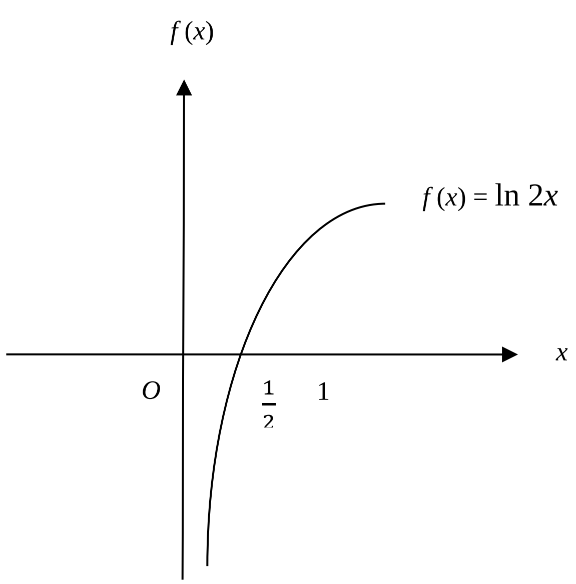
$\ln (999\,998) = \ln (5^t)$ [1 mark]

$\ln (999\,998) = t\ln 5$

$\frac{\ln 999\,998}{\ln 5} = t$ [1 mark]

8.58 minutes = t [1 mark]

7) (i) The function f is defined by $f: x \to \ln 2x$: $x \in \boldsymbol{R}$, $x > 0$.



correct orientation [1 mark]; passing through origin [1 mark]

(ii) $f: x \to \ln 2x$

Let $y = f(x)$

$y = \ln 2x$

$e^y = 2x$ [1 mark]

$e^x = 2y$ [1 mark]

$\frac{1}{2} e^x = f^{-1}(x)$ [1 mark]

(iii) Domain of f^{-1}, $x \in \boldsymbol{R}$ [1 mark]

Range of f^{-1}, $f(x) > 0$ [1 mark]

(iv) $fg(x) = f(\frac{1}{2}x - 4)$ [1 mark]

$= \ln[2(\frac{1}{2}x - 4)]$

$= \ln[x - 8]$ [1 mark]

8) $\frac{3-x}{2x-5} \times (2x-5)^2 \geq 0 \times (2x-5)^2$ [1 mark]

$(3 - x)(2x - 5) \geq 0$ [1 mark]

$\frac{5}{2} < \boldsymbol{x} \leq 3$ [1 mark]

9) $x + 4 = |2x|$

$x + 4 = +\sqrt{(2x)^2}$ [1 mark]

$(x + 4)^2 = (2x)^2$

$x^2 + 8x + 16 = 4x^2$

$0 = 3x^2 - 8x - 16$ [1 mark]

$0 = (x - 4)(3x + 4)$

$x = 4,\ x = -\frac{4}{3}$ [1+1 marks]

OR $x + 4 = |2x|$

$x + 4 = (+2x)$ or $x + 4 = (-2x)$ [1+1 marks]

$x = 4$ or $3x = -4$ [1 mark]

$x = -\frac{4}{3}$ [1 mark]