2014 Cape Unit 2 Paper 2 Solutions

1. (a) (i) $y = \ln(x^2+4) - x\tan^{-1}\left(\frac{x}{2}\right)$

$$\frac{dy}{dx} = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\left(\frac{1}{1+\left(\frac{x}{2}\right)^2}\right)$$
$$= \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\left(\frac{4}{4+x^2}\right)$$
$$= \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{x^2+4}$$
$$= -\tan^{-1}\left(\frac{x}{2}\right)$$

(ii) $x = a\cos^3 t$ $\quad$ $y = a\sin^3 t$

$\frac{dx}{dt} = -3a\sin t\cos^2 t$ $\quad$ $\frac{dy}{dt} = 3a\cos t\sin^2 t$

$$\frac{dy}{dx} = \frac{3a\cos t\sin^2 t}{-3a\sin t\cos t}$$
$$= -\tan t$$

$$y - a\sin^3 t = -\tan t\left(x - a\cos^3 t\right)$$
$$y - a\sin^3 t = -\frac{\sin t}{\cos t}\left(x - a\cos^3 t\right)$$
$$y\cos t - a\cos t\sin^3 t = -x\sin t + a\sin t\cos^3 t$$
$$y\cos t + x\sin t = a\cos t\sin^3 t + a\sin t\cos^3 t$$
$$= a\cos t\sin t\left(\sin^2 t + \cos^2 t\right)$$
$$= a\cos t\sin t$$

(b) (i) $\Delta = b^2 - 4ac = 9 - 4(1)(9)$

$= 9 - 36 < 0$ $\quad \therefore$ roots are not real

(ii) $x = \frac{-3+3\sqrt{3}i}{2} = \alpha$ $\quad$ $x = \frac{-3-3\sqrt{3}i}{2} = \beta$

$\arg\alpha = \pi - \tan^{-1}\left(\sqrt{3}\right) = \frac{2\pi}{3}$ $\quad$ $\arg\beta = -\pi + \tan^{-1}\left(\sqrt{3}\right) = -\frac{2\pi}{3}$

$|\alpha| = |\beta| = 3 \rightarrow \alpha = 3e^{i\left(\frac{2\pi}{3}\right)}$ $\quad$ $\beta = 3e^{i\left(-\frac{2\pi}{3}\right)}$

(iii)
$$\alpha^3 + \beta^3 = \left[3\left(e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}}\right)\right]^3$$
$$= 3^3\left(e^{i2\pi} + e^{-i2\pi}\right)$$
$$= 27\left(\cos 2\pi + i\sin 2\pi + \cos 2\pi - i\sin 2\pi\right)$$
$$= 27\left(2\cos 2\pi\right)$$
$$= 27(2)$$
$$= 54$$

(iv)
$$\alpha^3\beta^3 = (\alpha\beta)^3 = \left[9e^{i(2\pi-2\pi)}\right]^3$$
$$= 9^3(1)$$
$$= 729$$

eqn. $\Rightarrow x^2 - 54x + 729 = 0$

2. $F_n(x) = \int (\ln x)^n \, dx$

(i) $\int (\ln x)^n \, dx$

let $u = (\ln x)^n$ $\quad dv = 1$

$du = n(\ln x)^{n-1}\left(\frac{1}{x}\right)$ $\quad v = x$

$$\int (\ln x)^n \, dx = x(\ln x)^n - \int x\left(\frac{1}{x}\right)n(\ln x)^{n-1}\,dx$$
$$= x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$
$$= x(\ln x)^n - nF_{n-1}(x)$$

(ii)
$$F_3(x) = x(\ln x)^3 - 3F_2(x)$$
$$= x(\ln x)^3 - 3\left[x(\ln x)^2 - 2F_1(x)\right]$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6F_1(x)$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6\left[x\ln x - F_0\right]$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6F_0$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x$$

$$F_3(2) - F_3(1) = 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 12 - (0 - 0 + 0 - 6)$$
$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6$$

OR

$$F_3(2) - F_3(1) = 2(\ln 2)^3 - 3F_2(2) - \left[1(\ln 1)^3 - 3F_2(1)\right]$$
$$= 2(\ln 2)^3 - 3F_2(2) + 3F_2(1)$$
$$= 2(\ln 2)^3 - 3\left[2(\ln 2)^2 - 2F_1\right] + 3\left[1(\ln 1)^2 - 2F_1\right]$$
$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 6F_1$$

(b) (i)
$$\frac{y^2 + 2y + 1}{y^4 + 2y^2 + 1} = \frac{y^2 + 2y + 1}{\left(y^2 + 1\right)^2}$$
$$= \frac{Ay + B}{y^2 + 1} + \frac{Cy + D}{\left(y^2 + 1\right)^2}$$

$$y^2 + 2y + 1 = (Ay + B)\left(y^2 + 1\right) + Cy + D$$

y^3: $0 = A$

y^2: $1 = B$

y: $2 = A + C$; $\quad C = 2$

c: $1 = B + D$; $\quad D = 0$

$$\therefore \frac{y^2 + 2y + 1}{y^4 + 2y^2 + 1} = \frac{1}{y^2 + 1} + \frac{2y}{\left(y^2 + 1\right)^2}$$

(ii)
$$\int_0^1 \frac{y^2 + 2y + 1}{y^4 + 2y^2 + 1}\,dy = \int_0^1 \frac{1}{y^2 + 1}\,dy + \int_0^1 \frac{2y}{\left(y^2 + 1\right)^2}\,dy$$
$$= \left[\tan^{-1} y\right]_0^1 + \int_0^1 2y\left(y^2 + 1\right)dy$$

$$= \tan^{-1}y - \frac{1}{y^2+1}\Big|_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2} + 1$$

$$= \frac{\pi+2}{4}$$

3. (a) (i) $$\sum_{r=1}^{n} \frac{1}{2^r - 1} = 2 - \frac{1}{2^{n-1}}$$

$n = 1$: $\frac{1}{2^0} = 2 - \frac{1}{2^0}$

$1 = 1$

$n = k$: $\sum_{r=1}^{k} \frac{1}{2^r - 1} = 2 - \frac{1}{2^{k-1}}$

$n = k+1$: $\sum_{r=1}^{k+1} \frac{1}{2^r - 1} = 2 - \frac{1}{2^k}$

but $\sum_{r=1}^{k+1} \frac{1}{2^r - 1} = \sum_{r=1}^{k} \frac{1}{2^r - 1} + \frac{1}{2^k}$

$$= 2 - \frac{1}{2^{k-1}} + \frac{1}{2^k}$$

$$= 2 - \left(\frac{1}{2^k}\right)\frac{1}{2^{-1}} + \frac{1}{2^k}$$

$$= 2 - \left(\frac{1}{2^k}\right)(2-1)$$

$$= 2 - \frac{1}{2^k}$$

$\therefore$ if true for k, then true for all $n \in N$

(ii) $$\lim_{n\to\infty} S_n = 2 - \frac{1}{2^\infty - 1}$$

$$= 2 - \frac{1}{\infty}$$

$$= 2$$

(b) $f(x) = (1+x)^2 \sin x$

$f(0) = 0$

$f'(x) = 2(1+x)\sin x + (1+x)^2 \cos x$

$f'(0) = 1$

$f''(x) = 2\sin x + 2(1+x)\cos x + 2(1+x)\cos x - (1+x)^2 \sin x$

$f''(0) = 2 + 2 = 4$

$f'''(x) = 2\cos x + 4\cos x - 6(1+x)\sin x - (1+x)^2 \cos x$

$f'''(0) = 2 + 4 - 1 = 5$

$$\therefore f(x) = x + \frac{4x^2}{2!} + \frac{5x^3}{3!}$$

4. (a) (i) $\dfrac{{}^{20}C_7(2x)^{13}(3)^7}{{}^{20}C_6(2x)^{14}(3)^6} = \dfrac{38760}{77520}\cdot\dfrac{3}{2x} = \dfrac{3}{4x}$

(ii) (a) $(1+2x)^{10} = 1 + 10(2x) + \dfrac{10(9)(2x)^2}{2!}$

$= 1 + 20x + 180x^2$

(b) $(1.01)^{10} = (1 + 2(0.005))^{10}$

$= 1 + 20(0.005) + 180(0.005)^2$

$= 1 + 0.1 + 0.0045$

$= 1.1045$

(b) $\dfrac{n!}{(n-r)!r!} + \dfrac{n!}{(n-r+1)!(r-1)!} = \dfrac{(n+1)!}{(n-r+1)!r!}$

$(n-r+1)! = (n-r+1)(n-r)!$

$\dfrac{(n-r+1)!}{(n-r+1)} = (n-r)!$

$r! = r(r-1)!$

$\dfrac{r!}{r} = (r-1)!$

$\therefore \dfrac{n!}{\dfrac{(n-r+1)!}{(n-r+1)}r!} + \dfrac{n!}{(n-r+1)!\dfrac{r!}{r}} = \dfrac{n!(n-r+1)}{(n-r+1)!r!} + \dfrac{n!r}{(n-r+1)!r!}$

$= \dfrac{n!(n-r+r+1)}{(n-r+1)!r}$

$= \dfrac{n!(n+1)}{(n-r+1)!r!}$

$= \dfrac{(n+1)!}{(n-r+1)!r!}$

(c) (i) $f(1) = -1 + 3 + 4 = 6$

$f(3) = -27 + 9 + 4 = -14$

$f(1)f(3) < 0$

$\therefore$ by imvt a root exists in the interval

(ii) $f'(x) = -3x^2 + 3$

$x_2 = 2.1 - \dfrac{f(2.1)}{f'(2.1)}$

$= 2.20$

5. (a) (i) # of ways of arranging teams = 4!

# of ways of arranging teams among themselves = 2^5

$\therefore$ total # of arrangements = $4! \times 2^5$

= 768

(ii) (a) $P(R\cup B)=0.8$
$P(R)=0.4$
$P(B)=0.5$
$P(R\cap B)=?$

$$P(R\cup B)=P(R)+P(B)-P(R\cap B)$$
$$P(R\cap B)=P(R)+P(B)-P(R\cup B)$$
$$=0.4+0.5-0.8$$
$$=0.1$$

(b) $20\% \to 600$
$\therefore\ 100\% \to 5\times 20\% = 5\times 600 = 3000$

(b) (i) $A=\begin{pmatrix}1 & x & -1\\ 3 & 0 & 2\\ 2 & 1 & 0\end{pmatrix}$

$\det A \neq 0$
$1(0-2)-x(0-4)-1(3-0)\neq 0$
$-2+4x-3\neq 0$
$4x\neq 5$
$x\neq \frac{5}{4}$

(ii) $\det(AB)=\det(A)\det(B)=-21$
$\det B = 1(6-4)-2(4-4)+5(2-3)$
$=2-5=-3$

$\therefore\ \det A = 7$
$4x-5=7$
$4x=12$
$x=3$

(iii) cofactor of $A=\begin{pmatrix}-2 & 4 & 3\\ -1 & 2 & 5\\ 6 & -5 & -9\end{pmatrix}$

$$\text{adj } A=\begin{pmatrix}-2 & -1 & 6\\ 4 & 2 & -5\\ 3 & 5 & -9\end{pmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{7}\begin{pmatrix}-2 & -1 & 6\\ 4 & 2 & -5\\ 3 & 5 & -9\end{pmatrix}$$

6. (a) (i) $y' + y\tan x = \sec x$

IF: $e^{\int \tan x dx}$
$e^{-\ln(\cos x)}$
$\frac{1}{\cos x}$

$$y=\cos x\int \frac{1}{\cos x}\sec x dx$$

$$= \cos x \int \sec^2 x dx$$
$$= \cos x (\tan x + C)$$
$$= \sin x + C \cos x$$

(ii) $y = \sin x + C \cos x$

$$\frac{2}{\sqrt{2}} = \sin\frac{\pi}{4} + C\cos\frac{\pi}{4}$$
$$\frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}C$$
$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}C, \quad C = 1$$
$$\therefore y = \sin x + \cos x$$

(b) $y'' - 5y' = xe^{5x}$

CF: $m^2 - 5m = 0$

$$m(m-5) = 0$$
$$m = 0, 5$$
$$y = Ae^0 + Be^{5x}$$
$$y = A + Be^{5x}$$

$$y_p(x) = Ax^2e^{5x} + Bxe^{5x}$$
$$y_p'(x) = 2Axe^{5x} + 5Ax^2e^{5x} + Bxe^{5x} + 5Bxe^{5x}$$
$$y_p''(x) = 2Ae^{5x} + 10Axe^{5x} + 10Axe^{5x} + 25Ax^2e^{5x} + 5Be^{5x} + 5Be^{5x} + 25Bxe^{5x}$$

$$y'' - 5y' = 2Ae^{5x} + 10Axe^{5x} + 10Axe^{5x} + 25Ax^2e^{5x} + 5Be^{5x} + 5Be^{5x} + 25Bxe^{5x}$$
$$-10Axe^{5x} - 25Ax^2e^{5x} - 5Bxe^{5x} - 25Bxe^{5x} = xe^{5x}$$

$$2A + 10Ax + 5B = x$$
$$2A + 5B = 0 \ldots (1)$$
$$10A = 1 \ldots (2)$$

$$A = \frac{1}{10}$$

$$2\left(\frac{1}{10}\right) + 5B = 0$$
$$5B = -\frac{1}{5}$$
$$B = -\frac{1}{25}$$

$$\therefore y = \frac{1}{10}x^2e^{5x} - \frac{1}{25}xe^{5x}$$

Gen. soln.: $y = A + Be^{x} + \frac{1}{10}x^2e^{5x} - \frac{1}{25}xe^{5x}$