UNIT 1: ALGEBRA, GEOMETRY AND CALCULUS

MODULE 1: BASIC ALGEBRA AND FUNCTIONS

(A) Reasoning and Logic

Students should be able to:

1. identify simple and compound propositions;
2. establish the truth value of compound statements using truth tables;
3. state the converse, contrapositive and inverse of a conditional (implication) statement;
4. determine whether two statements are logically equivalent.

(B) The Real Number System — ℝ

Students should be able to:

1. perform binary operations;
2. use the concepts of identity, closure, inverse, commutativity, associativity, distributivity addition, multiplication and other simple binary operations;
3. perform operations involving surds;
4. construct simple proofs, specifically direct proofs, or proof by the use of counter examples;
5. establish simple proofs by using the principle of mathematical induction.

(C) Algebraic Operations

Students should be able to:

1. apply the Remainder Theorem;
2. use the Factor Theorem to find factors and to evaluate unknown coefficients;
3. extract all factors of \(a^n - b^n\) for positive integers \(n \leq 6\);
4. use the concept of identity of polynomial expressions.
(D) **Exponential and Logarithmic Functions**

Students should be able to:

1. define an exponential function \( y = a^x \) for \( a \in \mathbb{R} \);
2. sketch the graph of \( y = a^x \);
3. define a logarithmic function as the inverse of an exponential function;
4. define the exponential functions \( y = e^x \) and its inverse \( y = \ln x \), where \( \ln x = \log_a x \);
5. use the fact that \( y = \ln x \leftrightarrow x = e^y \);
6. simplify expressions by using laws of logarithms;
7. use logarithms to solve equations of the form \( a^x = b \);
8. solve problems involving changing of the base of a logarithm.

(E) **Functions**

Students should be able to:

1. define mathematically the terms: function, domain, range, one-to-one function (injective function), onto function (surjective function), many-to-one, one-to-one and onto function (bijective function), composition and inverse of functions;
2. prove whether or not a given simple function is one-to-one or onto and if its inverse exists;
3. use the fact that a function may be defined as a set of ordered pairs;
4. use the fact that if \( g \) is the inverse function of \( f \), then \( f(g(x)) = x \), for all \( x \), in the domain of \( g \);
5. illustrate by means of graphs, the relationship between the function \( y = f(x) \) given in graphical form and \( y = |f(x)| \) and the inverse of \( f(x) \), that is, \( y = f^{-1}(x) \).
(F) **The Modulus Function**

Students should be able to:

1. *define the modulus function;*

2. *use the properties:*
   
   (a) \( |x| \) *is the positive square root of* \( x^2 \);

   (b) \( |x| = |y| \) *if, and only if,* \( x^2 = y^2 \);

   (c) \( |x| < |y| \) *iff* \( -y < x < y \);

   (d) \( |x + y| \leq |x| + |y| \).

3. *solve equations and inequalities involving the modulus function, using algebraic or graphical methods.*

(G) **Cubic Functions and Equations**

Students should be able to use the relationship between the *sum of the roots, the product of the roots, the sum of the product of the roots pair-wise* and the coefficients of \( ax^3 + bx^2 + cx + d = 0 \).

**UNIT 1**

**MODULE 2: TRIGONOMETRY, GEOMETRY AND VECTORS**

(A) **Trigonometric Functions, Identities and Equations (all angles will be assumed to be in radians unless otherwise stated)**

Students should be able to:

1. *use compound-angle formulae;*

2. *use the reciprocal functions of sec \( x \), cosec \( x \) and cot \( x \);*

3. *derive identities for the following:*
   
   (a) \( \sin k\theta, \cos k\theta, \tan k\theta \), for \( k \in \mathbb{Q} \);

   (b) \( \tan^2 \theta, \cot^2 \theta,\sec^2 \theta \) and \( \csc^2 \theta \);

   (c) \( \sin A = \sin B, \cos A = \cos B \).

4. *prove further identities using Specific Objective 3;*

5. *express \( a \cos \theta + b \sin \theta \) in the form \( r \cos(\theta + \alpha) \) and \( r \sin(\theta + \alpha) \), where \( r \) is positive, \( 0 < \alpha < \frac{\pi}{2} \).
6. find the general solution of equations of the form:

(a) \( \sin k\theta = s, \)
(b) \( \cos k\theta = c, \)
(c) \( \tan k\theta = t, \)
(d) \( a\cos \theta + b\sin \theta = c, \)
    \[ \text{for } a, b, c, k, s, t \in \mathbb{R}; \]

7. find the solutions of the equations in Specific Objectives 6 above for a given range;

8. obtain maximum or minimum values of \( f(a\cos \theta + b\sin \theta) \) for \( 0 \leq \theta \leq 2\pi. \)

(B) Co-ordinate Geometry

Students should be able to:

1. find equations of tangents and normals to circles;
2. find the points of intersection of a curve with a straight line;
3. find the points of intersection of two curves;
4. obtain the Cartesian equation of a curve given its parametric representation;
5. obtain the parametric representation of a curve given its Cartesian equation;
6. determine the loci of points satisfying given properties.

(C) Vectors

Students should be able to:

1. express a vector in the form \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) or \( xi + yj + zk \)
   where \( i, j, k \) are unit vectors in directions of \( x-, y-, \) and \( z- \) axis respectively;
2. define equality of two vectors;
3. add and subtract vectors;
4. multiply a vector by a scalar quantity;
5. derive and use unit vectors, position vectors and displacement vectors;
6. find the magnitude and direction of a vector;
7. find the angle between two given vectors using scalar product;
8. find the equation of a line in vector form, parametric form, Cartesian form, given a point on the line and a vector parallel to the line;
9. determine whether two lines are parallel, intersecting, or skewed;
10. find the equation of the plane, in the form \( xi + yj + zk = d, \) \( \mathbf{r} \cdot \mathbf{n} = d, \) given a point in the plane and the normal to the plane.
UNIT 1
MODULE 3: CALCULUS I

(A) Limits

Students should be able to:

1. use graphs to determine the continuity and discontinuity of functions;

2. describe the behaviour of a function \( f(x) \) as \( x \) gets arbitrarily close to some given fixed number, using a descriptive approach;

3. use the limit notation \( \lim_{x \to a} f(x) = L \), \( f(x) \to L \) as \( x \to a \);

4. use the simple limit theorems:

   If \( \lim_{x \to a} f(x) = F \), \( \lim_{x \to a} g(x) = G \) and \( k \) is a constant,

   then \( \lim_{x \to a} k f(x) = kF \), \( \lim_{x \to a} f(x) g(x) = FG \), \( \lim_{x \to a} \{f(x) + g(x)\} = F + G \),

   and, provided \( G \neq 0 \), \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G} \);

5. use limit theorems in simple problems;

6. use the fact that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), demonstrated by a geometric approach;

7. identify the point(s) for which a function is (un)defined;

8. identify the points for which a function is continuous;

9. identify the point(s) where a function is discontinuous;

10. use the concept of left-handed or right-handed limit, and continuity.
Differentiation I

Students should be able to:

1. define the derivative of a function at a point as a limit;

2. differentiate, from first principles, functions such as:
   (a) \( f(x) = k \) where \( k \in \mathbb{R} \)
   (b) \( f(x) = x^n \), where \( n \in \{-3, -2, -1, -\frac{1}{2}, 1, 2, 3\} \)
   (c) \( f(x) = \sin x \)
   (d) \( f(x) = \cos x \)

3. use the sum, product and quotient rules for differentiation;

4. differentiate sums, products and quotients of:
   (a) polynomials,
   (b) trigonometric functions;

5. apply the chain rule in the differentiation of
   (a) composite functions (substitution),
   (b) functions given by parametric equations;

6. solve problems involving rates of change;

7. use the sign of the derivative to investigate where a function is increasing or decreasing;

8. apply the concept of stationary (critical) points;

9. calculate second derivatives;

10. interpret the significance of the sign of the second derivative;

11. use the sign of the second derivative to determine the nature of stationary points;

12. sketch graphs of polynomials, rational functions and trigonometric functions using the features of the function and its first and second derivatives (including horizontal and vertical asymptotes);

13. describe the behaviour of such graphs for large values of the independent variable;

14. obtain equations of tangents and normals to curves.
Integration I

Students should be able to:

1. recognise integration as the reverse process of differentiation;

2. demonstrate an understanding of the indefinite integral and the use of the integration notation \( \int f(x) \, dx \);

3. show that the indefinite integral represents a family of functions which differ by constants;

4. demonstrate use of the following integration theorems:
   (a) \( \int cf(x) \, dx = c \int f(x) \, dx \), where \( c \) is a constant,
   (b) \( \int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx \)

5. find:
   (a) indefinite integrals using integration theorems,
   (b) integrals of polynomial functions,
   (c) integrals of simple trigonometric functions;

6. integrate using substitution;

7. use the results:
   (a) \( \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(t) \, dt \),
   (b) \( \int_{0}^{a} f(x) \, dx = \int_{0}^{a} f(x-a) \, dx \) for \( a > 0 \),
   (c) \( \int_{a}^{b} f(x) \, dx = F(b) - F(a) \), where \( F'(x) = f(x) \);
The Unit is tested as follows

a) 3 SBA tests written by Harrison College teachers which together are worth 20% of the total mark. Each test covers one module.

b) Paper 1 Multiple Choice with 45 questions worth 30% of the total mark. The paper tests all three modules.

c) Paper 2 Long Answer with 6 questions (each with many parts) worth 50% of the total mark. The paper tests all three modules.
UNIT 2: COMPLEX NUMBERS, ANALYSIS AND MATRICES

MODULE 1: COMPLEX NUMBERS AND CALCULUS II

(A) Complex Numbers

Students should be able to:

1. recognise the need to use complex numbers to find the roots of the general quadratic equation \( ax^2 + bx + c = 0 \), when \( b^2 - 4ac < 0 \);

2. use the concept that complex roots of equations with constant coefficients occur in conjugate pairs;

3. write the roots of the equation in that case and relate the sums and products to \( a, b \) and \( c \);

4. calculate the square root of a complex number;

5. express complex numbers in the form \( a + bi \) where \( a, b \) are real numbers, and identify the real and imaginary parts;

6. add, subtract, multiply and divide complex numbers in the form \( a + bi \), where \( a \) and \( b \) are real numbers;

7. find the principal value of the argument \( \theta \) of a non-zero complex number, where \( -\pi < \theta \leq \pi \);

8. find the modulus and conjugate of a given complex number;

9. interpret modulus and argument of complex numbers on the Argand Diagram;

10. represent complex numbers, their sums, differences and products on an Argand diagram;

11. find the set of all points \( z \) (locus of \( z \)) on the Argand Diagram such that \( z \) satisfies given properties;

12. apply De Moivre's theorem for integral values of \( n \);

13. use \( e^{ix} = \cos x + i \sin x \), for real \( x \).
(B) Differentiation II

Students should be able to:

1. find the derivative of $e^{f(x)}$, where $f(x)$ is a differentiable function of $x$;
2. find the derivative of $\ln f(x)$ (to include functions of $x$ – polynomials or trigonometric);
3. apply the chain rule to obtain gradients and equations of tangents and normals to curves given by their parametric equations;
4. use the concept of implicit differentiation, with the assumption that one of the variables is a function of the other;
5. differentiate any combinations of polynomials, trigonometric, exponential and logarithmic functions;
6. differentiate inverse trigonometric functions;
7. obtain second derivatives, $f''(x)$, of the functions in 3, 4, 5 above.
8. find the first partial derivatives of $u = f(x, y)$ and $w = f(x, y, z)$;
9. find the second partial derivatives of $u = f(x, y)$ and $w = f(x, y, z)$.

(C) Integration II

Students should be able to:

1. express a rational function (proper and improper) in partial fractions in the cases where the denominators are:
   
   (a) distinct linear factors;
   (b) repeated linear factors;
(c) quadratic factors;

(d) repeated quadratic factors;

(e) combinations of (a) to (d) above (repeated factors will not exceed power 2);

2. express an improper rational function as a sum of a polynomial and partial fractions;

3. integrate rational functions in Specific Objectives 1 and 2 above;

4. integrate trigonometric functions using appropriate trigonometric identities;

5. integrate exponential functions and logarithmic functions;

6. find integrals of the form \( \int \frac{f'(x)}{f(x)} \, dx \);

7. use substitutions to integrate functions (the substitution will be given in all but the most simple cases);

8. use integration by parts for combinations of functions;

9. integrate inverse trigonometric functions;

10. derive and use reduction formulae to obtain integrals;

11. use the trapezium rule as an approximation method for evaluating the area under the graph of the function.

UNIT 2

MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS

(A) Sequences

Students should be able to:

1. define the concept of a sequence \( \{a_n\} \) of terms \( a_n \) as a function from the positive integers to the real numbers;

2. write a specific term from the formula for the \( n \)th term, or from a recurrence relation;

3. describe the behaviour of convergent and divergent sequences, through simple examples;

4. apply mathematical induction to establish properties of sequences.
Series

Students should be able to:

1. use the summation (\( \Sigma \)) notation;
2. define a series, as the sum of the terms of a sequence;
3. identify the \( n^{th} \) term of a series, in the summation notation;
4. define the \( m^{th} \) partial sum \( S_m \) as the sum of the first \( m \) terms of the sequence, that is, \( S_m = \sum_{r=1}^{m} a_r \);
5. apply mathematical induction to establish properties of series;
6. find the sum to infinity of a convergent series;
7. apply the method of differences to appropriate series, and find their sums;
8. use the Maclaurin theorem for the expansion of series;
9. use the Taylor theorem for the expansion of series.

The Binomial Theorem

Students should be able to:

1. explain the meaning and use simple properties of \( n! \) and \( \binom{n}{r} \), that is, \( n \ C_r \), where \( n, r \in \mathbb{Z} \);
2. recognise that \( n \ C_r \) that is, \( \binom{n}{r} \), is the number of ways in which \( r \) objects may be chosen from \( n \) distinct objects;
3. expand \( (a + b)^n \) for \( n \in \mathbb{Q} \);
4. apply the Binomial Theorem to real-world problems, for example, in mathematics of finance, science.

Roots of Equations

Students should be able to:

1. test for the existence of a root of \( f(x) = 0 \) where \( f \) is continuous using the Intermediate Value Theorem;
2. use interval bisection to find an approximation for a root in a given interval;
3. use linear interpolation to find an approximation for a root in a given interval;
4. explain, in geometrical terms, the working of the Newton-Raphson method;
5. use the Newton-Raphson method to find successive approximations to the roots of \( f(x) = 0 \), where \( f \) is differentiable;
6. use a given iteration to determine a root of an equation to a specified degree of accuracy.
UNIT 2
MODULE 3: COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS

(A) Counting

Students should be able to:

1. state the principles of counting;
2. find the number of ways of arranging \( n \) distinct objects;
3. find the number of ways of arranging \( n \) objects some of which are identical;
4. find the number of ways of choosing \( r \) distinct objects from a set of \( n \) distinct objects;
5. identify a sample space;
6. identify the numbers of possible outcomes in a given sample space;
7. use Venn diagrams to illustrate the principles of counting;
8. use possibility space diagram to identify a sample space;
9. define and calculate \( P(A) \), the probability of an event \( A \) occurring as the number of possible ways in which \( A \) can occur divided by the total number of possible ways in which all equally likely outcomes, including \( A \), occur;
10. use the fact that \( 0 \leq P(A) \leq 1 \);
11. demonstrate and use the property that the total probability for all possible outcomes in the sample space is 1;
12. use the property that \( P(A') = 1 - P(A) \) is the probability that event \( A \) does not occur;
13. use the property \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \) for event \( A \) and \( B \);
14. use the property \( P(A \cap B) = 0 \) or \( P(A \cup B) = P(A) + P(B) \), where \( A \) and \( B \) are mutually exclusive events;
15. use the property \( P(A \mid B) = P(A) \times P(B) \), where \( A \) and \( B \) are independent events;
16. use the property \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \), where \( P(B) \neq 0 \);
17. use a tree diagram to list all possible outcomes for conditional probability.

(B) Matrices and Systems of Linear Equations

Students should be able to:

1. operate with conformable matrices, carry out simple operations and manipulate matrices using their properties;
2. evaluate the determinants of \( n \times n \) matrices, \( 1 \leq n \leq 3 \);
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(C) Differential Equations and Modelling

Students should be able to:

1. solve first order linear differential equations $y' - ky = f(x)$ using an integrating factor, given that $k$ is a real constant or a function of $x$, and $f$ is a function;

2. solve first order linear differential equations given boundary conditions;

3. solve second order ordinary differential equations with constant coefficients of the form

$$ay'' + by' + cy' = 0 = f(x),$$

where $a, b, c \in \mathbb{R}$ and $f(x)$ is:

(a) a polynomial,
(b) an exponential function,
(c) a trigonometric function;

and the complementary function may consist of

(a) 2 real and distinct roots;
(b) 2 equal roots;
(c) 2 complex roots.

4. solve second order ordinary differential equation given boundary conditions;

5. use substitution to reduce a second order ordinary differential equation to a suitable form.