CAPE PURE MATHEMATICS

UNIT 1

♦ UNIT 1: ALGEBRA, GEOMETRY AND CALCULUS

MODULE 1: BASIC ALGEBRA AND FUNCTIONS

(A) Reasoning *and Logic*

Students should be able to:

1. *identify simple and compound propositions;*
2. *establish the truth value of compound statements using truth tables;*
3. *state the converse, contrapositive and inverse of a conditional (implication) statement;*
4. *determine whether two statements are logically equivalent.*

(B) The Real Number System – $\mathbb{R}$

Students should be able to:

1. *perform binary operations;*
2. *use the concepts of identity, closure, inverse, commutativity, associativity, distributivity addition, multiplication and other simple binary operations;*
3. *perform operations involving surds;*
4. construct simple proofs, specifically direct proofs, or proof by the use of counter examples;
5. establish simple proofs by using the principle of mathematical induction.

(C) *Algebraic Operations*

Students should be able to:

1. *apply the Remainder Theorem;*
2. use the Factor Theorem to find factors and to evaluate unknown coefficients;
3. extract all factors of $a^n - b^n$ for positive integers $n \leq 6$;
4. use the concept of identity of polynomial expressions.

(D) *Exponential and Logarithmic Functions*

Students should be able to:

1. *define an exponential function* $y = a^x$ *for* $a \in \mathbb{R}$;
2. *sketch the graph of* $y = a^x$;
3. *define a logarithmic function as the inverse of an exponential function;*
4. *define the exponential functions* $y = e^x$ *and its inverse* $y = \ln x$, *where* $\ln x \equiv \log_e x$;
5. *use the fact that* $y = \ln x \Leftrightarrow x = e^y$;
6. *simplify expressions by using laws of logarithms;*
7. *use logarithms to solve equations of the form* $a^x = b$;
8. solve problems involving changing of the base of a logarithm.

(E) Functions

Students should be able to:

1. *define mathematically the terms: function, domain, range, one-to-one function (injective function), onto function (surjective function), many-to-one, one-to-one and onto function (bijective function), composition and inverse of functions;*
2. *prove whether or not a given simple function is one-to-one or onto and if its inverse exists;*
3. *use the fact that a function may be defined as a set of ordered pairs;*
4. *use the fact that if g is the inverse function of f, then* $f[g(x)] = x$, *for all* x, *in the domain of g;*
5. illustrate by means of graphs, the relationship between the function $y = f(x)$ given in graphical form and $y = |f(x)|$ and the inverse of $f(x)$, that is, $y = f^{-1}(x)$.

(F) ***The Modulus Function***

Students should be able to:

1. *define the modulus function;*

2. *use the properties:*

   (a) $|x|$ *is the positive square root of* x^2 *;*

   (b) $|x| < |y|$ *if, and only if,* $x^2 < y^2$*;*

   (c) $|x| < |y| \Leftrightarrow$ *iff* $-y < x < y$ *;*

   (d) $|x + y| \leq |x| + |y|$.

3. *solve equations and inequalities involving the modulus function, using algebraic or graphical methods.*

(G) ***Cubic Functions and*** **Equations**

Students should be able to use the relationship between the *sum of the roots, the product of the roots, the sum of the product of the roots pair-wise* and the coefficients *of* $ax^3 + bx^2 + cx + d = 0$.

UNIT 1
MODULE 2: *TRIGONOMETRY,* GEOMETRY AND VECTORS

(A) Trigonometric Functions, Identities and Equations (all angles will be assumed to be in radians unless otherwise stated)

Students should be able to*:*

1. *use compound-angle formulae;*

2. *use the reciprocal functions of* sec x, cosec x *and* cot x*;*

3. *derive identities for the following:*

   (a) sin kA, cos kA, tan kA, *for* $k \in \mathbb{Q}$;

   (b) $\tan^2 x$, $\cot^2 x$, $\sec^2 x$ and $\text{cosec}^2 x$*;*

   (c) sin $A \pm$ sin B, cos $A \pm$ cos B.

4. *prove further identities using Specific Objective 3;*

5. express $a\cos\theta + b\sin\theta$ in the form $r\cos(\theta \pm \alpha)$ and $r\sin(\theta \pm \alpha)$, where r is positive, $0 < \alpha < \frac{\pi}{2}$;

6. find the general solution of equations of the form:

   (a) $\sin k\theta = s$,

   (b) $\cos k\theta = c$,

   (c) $\tan k\theta = t$,

   (d) $a \cos \theta + b \sin \theta = c$,
   for $a, b, c, k, s, t, \in \mathbb{R}$;

7. *find the solutions of the equations in Specific Objectives* 6 *above for a given range;*

8. obtain maximum or minimum values of f $(a \cos \theta + b \sin \theta)$ for $0 \leq \theta \leq 2\pi$.

(B) Co-ordinate Geometry

Students should be able to:

1. find equations of tangents and normals to circles;

2. find the points of intersection of a curve with a straight line;

3. find the points of intersection of two curves;

4. obtain the Cartesian equation of a curve given its parametric representation;

5. obtain the parametric representation of a curve given its Cartesian equation;

6. *determine the loci of points satisfying given properties.*

(C) Vectors

Students should be able to:

1. *express a vector in the form* $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *or* $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

   where **i**, **j** *and* **k** *are unit vectors in directions of* x-,y- *and* z-*axis respectively;*

2. define equality of two vectors;

3. add and subtract vectors;

4. multiply a vector by a scalar quantity;

5. *derive and use unit vectors, position vectors and displacement vectors;*

6. find the magnitude and direction of a vector;

7. *find the angle between two given vectors using scalar product;*

8. *find the equation of a line in vector form, parametric form, Cartesian form, given a point on the line and a vector parallel to the line;*

9. *determine whether two lines are parallel, intersecting, or skewed;*

10. *find the equation of the plane, in the form* $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = d$, $\mathbf{r}.\mathbf{n} = d$, *given a point in the plane and the normal to the plane.*

UNIT 1
MODULE 3: CALCULUS I

(A) Limits

Students should be able to:

1. use graphs to determine the continuity and discontinuity of functions;

2. describe the behaviour of a function f (x) as x gets arbitrarily close to some given fixed number, using a descriptive approach;

3. use the limit notation $\lim_{x \to a} \mathrm{f}(x) = L, \mathrm{f}(x) \to \mathrm{L}$ as $x \to a$;

4. *use the simple limit theorems*:
   If $\lim_{x \to a} \mathrm{f}(x) = F$, $\lim_{x \to a} \mathrm{g}(x) = G$ and k is a constant,

   then $\lim_{x \to a} k\mathrm{f}(x) = kF$, $\lim_{x \to a} \mathrm{f}(x)\mathrm{g}(x) = FG$, $\lim_{x \to a} \{\mathrm{f}(x) + \mathrm{g}(x)\} = F + G$,

   and, provided $G \neq 0$, $\lim_{x \to a} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \frac{F}{G}$;

5. *use limit theorems in simple problems;*

6. *use the fact that* $\lim_{x \to 0} \frac{\sin x}{x} = 1$, *demonstrated by a geometric approach;*

7. *identify the point(s) for which a function is (un)defined;*

8. identify the points for which a function is continuous;

9. *identify the point(s) where a function is discontinuous;*

10. *use the concept of left-handed or right-handed limit, and continuity.*

(B) Differentiation I

Students should be able to:

1. *define the derivative of a function at a point as a limit;*

2. *differentiate, from first principles, functions such as*:

   (a) $f(x) = k$ *where* $k \in \mathbb{R}$

   (b) $f(x) = x^n$, *where* $n \in \{-3, -2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2, 3\}$,

   (c) $f(x) = \sin x$,

   (d) $f(x) = \cos x$.

3. use the *sum*, product and quotient rules for differentiation;

4. differentiate *sums*, products and quotients of:

   (a) polynomials,

   (b) trigonometric functions*;*

5. *apply the chain rule in the differentiation of*

   (a) *composite functions (substitution),*

   (b) *functions given by parametric equations;*

6. *solve problems involving rates of change;*

7. use the sign of the derivative to investigate where a function is increasing or decreasing;

8. *apply the concept of stationary (critical) points;*

9. calculate second derivatives;

10. interpret the significance of the sign of the second derivative;

11. use the sign of the second derivative to determine the nature of stationary points;

12. *sketch graphs of polynomials, rational functions and trigonometric functions using the features of the function and its first and second derivatives (including horizontal and vertical asymptotes);*

13. describe the behaviour of such graphs for large values of the independent variable;

14. obtain equations of tangents and normals to curves.

(C) Integration I

Students should be able to:

1. *recognise integration as the reverse process of differentiation;*

2. demonstrate an understanding of the indefinite integral and the use of the integration notation $\int \mathrm{f}(x)\,\mathrm{d}x$;

3. show that the indefinite integral represents a family of functions which differ by constants;

4. demonstrate use of the following integration theorems:

   (a) $\int c\mathrm{f}(x)\mathrm{d}x = c\int \mathrm{f}(x)\mathrm{d}x$, where c is a constant,

   (b) $\int \{\mathrm{f}(x) \pm \mathrm{g}(x)\}\,\mathrm{d}x = \int \mathrm{f}(x)\,\mathrm{d}x \pm \int \mathrm{g}(x)\,\mathrm{d}x$

5. find:

   (a) indefinite integrals using integration theorems,

   (b) integrals of polynomial functions,

   (c) integrals of simple trigonometric functions;

6. integrate *using substitution;*

7. use the results:

   (a) $\int_a^b \mathrm{f}(x)\mathrm{d}x = \int_a^b \mathrm{f}(t)\mathrm{d}t$,

   (b) $\int_0^a \mathrm{f}(x)\mathrm{d}x = \int_0^a \mathrm{f}(x-a)\mathrm{d}x$ for $a > 0$,

   (c) $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$, where $\mathrm{F}'(x) = \mathrm{f}(x)$;

8. apply *integration to:*

   (a) finding areas under the curve,

   (b) finding areas between two curves,

   (c) *finding volumes of revolution by rotating regions about both the x- and y-axes,*

9. *given a rate of change with or without initial boundary conditions:*

   (a) *formulate a differential equation of the form $y' = \mathrm{f}(x)$ or $y'' = \mathrm{f}(x)$ where f is a polynomial or a trigonometric function,*

   (b) *solve the resulting differential equation in* (a) *above and interpret the solution where applicable.*

The Unit is tested as follows

a) 3 SBA tests written by Harrison College teachers which together are worth 20% of the total mark. Each test covers one module.

b) Paper 1 Multiple Choice with 45 questions worth 30% of the total mark. The paper tests all three modules.

c) Paper 2 Long Answer with 6 questions (each with many parts) worth 50% of the total mark. The paper tests all three modules.

CAPE PURE MATHEMATICS

UNIT 2

♦ UNIT 2: *COMPLEX NUMBERS*, ANALYSIS AND MATRICES

MODULE 1: *COMPLEX NUMBERS AND CALCULUS II*

(A) Complex Numbers

Students should be able to:

1. recognise the need to use complex numbers to find the roots of the general quadratic equation $ax^2 + bx + c = 0$, when $b^2 - 4ac < 0$;

2. *use the concept that complex roots of equations with constant coefficients occur in conjugate pairs*;

3. write the roots of the equation in that case and relate the sums and products to a, b and c;

4. calculate the square root of a complex number;

5. express complex numbers in the form $a + b\mathrm{i}$ where a, b are real numbers, and identify the real and imaginary parts;

6. add, subtract, multiply and divide complex numbers in the form $a + b\mathrm{i}$, where a and b are real numbers;

7. find the principal value of the argument θ of a non-zero complex number, where $-\pi < \theta \leq \pi$;

8. find the modulus and conjugate of a given complex number;

9. interpret modulus and argument of complex numbers on the Argand Diagram;

10. represent complex numbers, their sums, differences and products on an Argand diagram;

11. find the set of all points z (locus of z) on the Argand Diagram such that z satisfies given properties;

12. apply De Moivre's theorem for integral values of n;

13. *use* $e^{ix} = \cos x + i \sin x$, *for real* x.

(B) Differentiation II

Students should be able to:

1. find the derivative of $e^{f(x)}$, where f (x) is a differentiable function of x;

2. find the derivative of ln f (x) (to include functions of x – polynomials or trigonometric);

3. apply the chain rule to obtain gradients *and equations* of tangents and normals to curves given by their parametric equations;

4. use the concept of implicit differentiation, with the assumption that one of the variables is a function of the other;

5. differentiate any combinations of polynomials, trigonometric, exponential and logarithmic functions;

6. differentiate inverse trigonometric functions;

7. obtain second derivatives, f $''(x)$, of the functions in 3, 4, 5 above.

8. *find the first partial derivatives of* $u = f(x, y)$ *and* $w = f(x, y, z)$;

9. *find the second partial derivatives of* $u = \mathrm{f}(x, y)$ *and* $w = \mathrm{f}(x, y, z)$.

(C) Integration II

Students should be able to:

1. express a rational function ***(proper and improper)*** in partial fractions in the cases where the denominators are:

   (a) distinct linear factors;

   (b) repeated linear factors;

(c) quadratic factors;

(d) repeated quadratic factors;

(e) combinations of (a) to (d) above (repeated factors will not exceed power 2);

2. express an improper rational function as a sum of a polynomial and partial fractions;

3. integrate rational functions in Specific Objectives 1 and 2 above;

4. integrate trigonometric functions using appropriate trigonometric identities;

5. integrate exponential functions and logarithmic functions;

6. find integrals of the form $\int \frac{f'(x)}{f(x)} dx$;

7. use substitutions to integrate functions (the substitution will be given in all but the most simple cases);

8. use integration by parts for combinations of functions;

9. integrate inverse trigonometric functions;

10. derive and use reduction formulae to obtain integrals;

11. use the trapezium rule as an approximation method for evaluating the area under the graph of the function.

UNIT 2
MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS

(A) Sequences

Students should be able to:

1. define the concept of a sequence $\{a_n\}$ of terms a_n as a function from the positive integers to the real numbers;

2. write a specific term from the formula for the nth term, or from a recurrence relation;

3. describe the behaviour of convergent and divergent sequences, through simple examples;

4. apply mathematical induction to establish properties of sequences.

(B) Series

Students should be able to:

1. use the summation (Σ) notation;

2. define a series, as the sum of the terms of a sequence;

3. identify the nth term of a series, in the summation notation;

4. define the mth partial sum S_m as the sum of the first m terms of the sequence, that is, $S_m = \sum_{r=1}^{m} a_r$;

5. apply mathematical induction to establish properties of series;

6. find the sum to infinity of a convergent series;

7. apply the method of differences to appropriate series, and find their sums;

8. use the Maclaurin theorem for the expansion of series;

9. *use the Taylor theorem for the expansion of series.*

(C) The Binomial Theorem

Students should be able to:

1. explain the meaning and use simple properties of $n!$ and $\binom{n}{r}$, that is, ${}^{n}C_{r}$, where $n, r \in \mathbb{Z}$;

2. *recognise that* ${}^{n}C_{r}$ *that is,* $\binom{n}{r}$, *is the number of ways in which r objects may be chosen from n distinct objects;*

3. expand $(a+b)^n$ for $n \in \mathbb{Q}$;

4. apply the Binomial Theorem to real-world problems, for example, in mathematics of finance, science.

(D) Roots of Equations

Students should be able to:

1. test for the existence of a root of $f(x) = 0$ where f is continuous using the Intermediate Value Theorem;

2. *use interval bisection to find an approximation for a root in a given interval;*

3. *use linear interpolation to find an approximation for a root in a given interval;*

4. explain, in geometrical terms, the working of the Newton-Raphson method;

5. use the Newton-Raphson method to find successive approximations to the roots of $f(x) = 0$, where f is differentiable;

6. *use a given iteration to determine a root of an equation to a specified degree of accuracy.*

UNIT 2

MODULE 3: COUNTING, MATRICES AND *DIFFERENTIAL EQUATIONS*

(A) Counting

Students should be able to:

1. state the principles of counting;

2. find the number of ways of arranging n distinct objects;

3. find the number of ways of arranging n objects some of which are identical;

4. find the number of ways of choosing r distinct objects from a set of n distinct objects;

5. identify a sample space;

6. *identify* the numbers of possible outcomes in a given sample space;

7. *use Venn diagrams to illustrate the principles of counting;*

8. *use possibility space diagram to identify a sample space*;

9. *define* and calculate $\mathrm{P}(A)$, the probability of an event A occurring as the number of possible ways in which A can occur divided by the total number of possible ways in which all equally likely outcomes, including A, occur;

10. use the fact that $0 \leq \mathrm{P}(A) \leq 1$;

11. demonstrate and use the property that the total probability for all possible outcomes in the sample space is 1;

12. use the property that $\mathrm{P}(A') = 1 - \mathrm{P}(A)$ is the probability that event A does not occur;

13. use the property $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ for event A and B;

14. use the property $\mathrm{P}(A \cap B) = 0$ or $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$, where A and B are mutually exclusive events;

15. use the property $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$, where A and B are independent events;

16. *use the property* $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ *where* $\mathrm{P}(B) \neq 0$.

17. *use a tree diagram to list all possible outcomes for conditional probability.*

(B) Matrices and Systems of Linear Equations

Students should be able to:

1. operate with conformable matrices, carry out simple operations and manipulate matrices using their properties;

2. evaluate the determinants of $n \times n$ matrices, $1 \leq n \leq 3$;

3. reduce a system of linear equations to echelon form;

4. row-reduce the augmented matrix of an $n \times n$ system of linear equations, $n = 2, 3$;

5. determine whether the system is consistent, and if so, how many solutions it has;

6. find all solutions of a consistent system;

7. invert a non-singular 3×3 matrix;

8. solve a 3×3 system of linear equations, having a non-singular coefficient matrix, by using its inverse.

(C) ***Differential Equations and Modeling***

Students should be able to:

1. *solve first order linear differential equations* $y' - ky = \mathrm{f}(x)$ *using an integrating factor, given that* k *is a real constant or a function of* x*, and* f *is a function;*

2. *solve first order linear differential equations given boundary conditions;*

3. *solve second order ordinary differential equations with constant coefficients of the form*

$$ay'' + by' + cy = 0 = \mathrm{f}(x), \text{ where } a, b, c \in \mathbb{R} \text{ and } \mathrm{f}(x) \text{ is:}$$

   (a) *a polynomial,*
   (b) *an exponential function,*
   (c) *a trigonometric function;*

   and the complementary function may consist of

   (a) *2 real and distinct roots;*
   (b) *2 equal roots;*
   (c) *2 complex roots.*

4. *solve second order ordinary differential equation given boundary conditions;*

5. *use substitution to reduce a second order ordinary differential equation to a suitable form.*

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